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HARMONIC MEASURES, HAUSDORFF MEASURES AND POSITIVE EIGENFUNCTIONS

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Abstract

Let M be a compact negatively curved Riemannian manifold with universal covering $\tilde{M}$, and let $\delta_0 > 0$ be the negative of the bottom of the positive spectrum of the Laplacean Δ on $\tilde{M}$. We use methods from ergodic theory to show that $\Delta + \delta_0$ admits a Green's function which decays exponentially with the distance. Moreover for almost every point $\zeta \in \partial\tilde{M}$ with respect to a suitable Borel-measure which is positive on open sets, the unique minimal positive $\Delta + \delta_0 - \epsilon$-harmonic functions on $\tilde{M}$ with pole at ζ normalized at a point $x \in \tilde{M}$ converge as $\epsilon \to 0$ uniformly on compact sets to a minimal positive $\Delta + \delta_0$-harmonic function.

1. Introduction

Let M be an n-dimensional compact manifold of negative sectional curvature, and let $\tilde{M}$ be its universal covering. For every $x \in \tilde{M}$ the harmonic measure ω^x at x is a Borel-probability measure on the ideal boundary $\partial\tilde{M}$ of $\tilde{M}$, which via the canonical identification can be viewed as a measure on the fibre $T^1_x\tilde{M}$ at x of the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$.

Let Γ be the fundamental group of M acting as a group of isometries on $\tilde{M}$ and $T^1\tilde{M}$. For $\Psi \in \Gamma$ we then have $\omega^{\Psi x} = \omega^x \circ (d\Psi)^{-1}$, and hence the measures ω^x can be transported to measures on the fibres of the unit tangent bundle T^1M of M.

Denote by DTM (resp. $DT\tilde{M}$) the smooth fibre bundle over M (resp. $\tilde{M}$) whose fibre DTM_x at $x \in M$ (resp. $DT\tilde{M}_x$ at $x \in \tilde{M}$) equals $T^1_xM \times T^1_xM$ (resp. $T^1_x\tilde{M} \times T^1_x\tilde{M}$). We call a function β on DTM *symmetric* if β is invariant under the natural involution $(v, w) \to (w, v)$. In Section 2 of this note we show:

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Theorem A. *There is a Hölder-continuous symmetric function* $\delta\colon DTM \to [0,\infty)$ *with the following properties:*

1) *There is a number* $\kappa > 0$ *such that for every* $x \in M$ *the restriction of* δ^κ *to* DTM_x *is a quasi-distance on* $T^1_x M$ *defining the usual topology.*
2) *For every* $x \in M$ *the measure* ω^x *is the* $1/\kappa-$*dimensional spherical measure on* $T^1_x M$ *induced by* δ^κ.

Denote by Δ the Laplacean on $\tilde{M}$, and let $\delta_0 > 0$ be the negative of the bottom of the positive spectrum of Δ on $\tilde{M}$, which equals the top of the spectrum of Δ acting on square-integrable functions on $\tilde{M}$ (see [21]). For every $\epsilon > 0$ the differential operator $\Delta_\epsilon = \Delta + \delta_0 - \epsilon$ is *weakly coercive* in the sense of Ancona [1], and hence the *Martin boundary* of Δ_ϵ can naturally be identified with the ideal boundary $\partial\tilde{M}$ of $\tilde{M}$ (see [1]). In other words, Δ_ϵ admits a *Green's function* G_ϵ on $\tilde{M} \times \tilde{M} - \{(x,x) \mid x \in \tilde{M}\}$, and the *Martin kernel* K_ϵ of Δ_ϵ is a Hölder-continuous function on $\tilde{M} \times \tilde{M} \times \partial\tilde{M}$ such that for every $x \in \tilde{M}$ and every $\zeta \in \partial\tilde{M}$ the assignment $y \to K_\epsilon(x,y,\zeta)$ is the unique minimal positive Δ_ϵ-harmonic function on $\tilde{M}$ with *pole* at ζ, which is normalized to be 1 at x. Since Δ_ϵ is in fact *coercive* the results of Ancona imply that there are numbers $c_\epsilon > 0, \chi_\epsilon > 0$ such that $G_\epsilon(x,y) \leq c_\epsilon e^{-\chi_\epsilon \operatorname{dist}(x,y)}$ whenever the distance $\operatorname{dist}(x,y)$ of $x,y \in \tilde{M}$ is not smaller than 1.

The operator $\Delta_0 = \Delta + \delta_0$ fails to be weakly coercive in the sense of Ancona. In fact, Ancona gave an example of a simply connected manifold $\tilde{N}_1$ of bounded negative curvature for which Δ_0 does not even admit a Green's function [2]. Ancona also constructed a simply connected manifold $\tilde{N}_2$ of bounded negative curvature such that Δ_0 admits a Green's function, but the Martin boundary of Δ_0 consists of a unique point. However, under our assumption that $\tilde{M}$ is the universal covering of a compact manifold, these cases can not occur. More precisely, we denote for $p \in \tilde{M}$ and $R > 0$ by $S(p,R)$ the distance sphere of radius R about p in $\tilde{M}$, and let $\lambda_{p,R}$ be the Lebesgue measure on $S(p,R)$ induced by the restriction of the Riemannian metric on $\tilde{M}$ to $S(p,R)$. In Section 3 and Section 5 we show

Theorem B. *Assume that* $\tilde{M}$ *is the universal covering of a compact manifold* M. *Then the operator* $\Delta + \delta_0$ *admits a Green's function* G_0 *with the following properties:*

1) *There are constants* $a > 0, \chi > 0$ *such that* $G_0(x,y) \leq a e^{-\chi \operatorname{dist}(x,y)}$ *for all* $x,y \in \tilde{M}$ *with* $\operatorname{dist}(x,y) \geq 1$.
2) *There is a number* $c > 0$ *such that* $\int_{S(p,R)} G_0(p,y)^2 d\lambda_{p,R}(y) \leq c$ *for all* $p \in \tilde{M}, R \geq 1$.
3) $\liminf_{R\to\infty} \int_{S(p,R)} G_0(p,y)^{2-\epsilon} d\lambda_{p,R}(y) = \infty$ *for every* $\epsilon > 0$.

Moreover we obtain in Section 5:

Theorem C. *There is a $\pi_1(M)$-invariant measure class $\nu(\infty)$ on $\partial\tilde{M}$ such that for $\nu(\infty)$-almost every $\zeta \in \partial\tilde{M}$ and every $x \in \tilde{M}$ the functions $y \to K_\epsilon(x,y,\zeta)$ converge as $\epsilon \to 0$ uniformly on compact subsets of $\tilde{M}$ to a minimal positive Δ_0-harmonic function on $\tilde{M}$.*

Recall that δ_0 equals the infimum of the Rayleigh-quotients $\int \|\nabla\phi\|^2\, dx / \int \phi^2\, dx$ over all nontrivial smooth functions ϕ on $\tilde{M}$ with compact support. However δ_0 can also be expressed via a variational equation on the unit tangent bundle T^1M of M. For its formulation recall that the *geodesic flow* Φ^t is a smooth dynamical system on T^1M, generated by the *geodesic spray* X. There is a Hölder-continuous Φ^t-invariant decomposition $TT^1M = \mathbb{R}X \oplus TW^{ss} \oplus TW^{su}$ where TW^{ss} (resp. TW^{su}) is the tangent bundle of the *strong stable foliation* W^{ss} (resp. the *strong unstable foliation* W^{su}). The leaves of the *stable foliation* W^s with tangent bundle $TW^s = \mathbb{R}X \oplus TW^{ss}$ are smoothly immersed submanifolds of T^1M which are mapped by the canonical projection $P\colon T^1M \to M$ locally diffeomorphically onto M. Thus the Riemannian metric on M induces a Riemannian metric g^s on TW^s and a family λ^s of Lebesgue measures on the leaves of W^s. Write also $\langle , \rangle$ instead of g^s .

The *stable Laplacean* Δ^s is a second order differential operator on T^1M with Hölder continuous coefficients. For a smooth function ϕ on T^1M the value of $\Delta^s\phi$ at $v \in T^1M$ just equals the value at v of the Laplacean of the Riemannian manifold $(W^s(v), g^s)$ applied to the restriction of ϕ to the leaf $W^s(v)$ of W^s through v. Moreover denote the gradient of $\phi|(W^s(v), g^s)$ at v by $(\nabla^s\phi)(v) \in T_vW^s$.

Let η be a Borel-probability measure on T^1M which is absolutely continuous with respect to the stable and strong unstable foliation, with conditionals on stable manifolds in the Lebesgue measure class. Recall from [12] the definition of the g^s - *gradient* of η (if this exists). It is the unique section Y of TW^s which satisfies

$$\int \phi(\Delta^s + Y)(\psi)\, d\eta = \int \psi(\Delta^s + Y)(\phi)\, d\eta$$

for all smooth functions ϕ, ψ on T^1M.

Call a section Z of TW^s *of class* $C_s^{1,\alpha}$ for some $\alpha > 0$ if Z is Hölder-continuous of class α and differentiable along the leaves of the stable foliation, with leafwise first order jets of class C^α. If Z is of class $C_s^{1,\alpha}$, then for every $v \in T^1M$ the divergence $\operatorname{div} Z(v)$ of $Z|(W^s(v), \lambda^s)$ is defined at v and the assignment $v \to \operatorname{div} Z(v)$ is of class C^α.

With thise notation in Section 4 of this note we show

Theorem D. *Let η be a Borel-probability measure on T^1M, which is absolutely continuous with respect to the stable and unstable foliations, with conditionals on stable manifolds in the Lebesgue measure class. Assume that the g^s-gradient Y of η is of class $C_s^{1,\alpha}$ for some $\alpha > 0$. Then*

$$-\delta_0 = \sup\{\int \phi(\Delta^s(\phi) + Y(\phi) + \phi[\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2])\, d\eta \mid \phi \in C^\infty(T^1M), \int \phi^2\, d\eta = 1\}.$$

As a corollary, we find a new proof of a result of Ledrappier; namely, let σ be the unique Borel-probability measure on T^1M such that $\int(\Delta^s\phi)\, d\sigma = 0$ for every smooth function ϕ on T^1M (see [18], [12]). The g^s-gradient Y of σ satisfies div $(Y) = -\|Y\|^2$, and $\int \|Y\|^2\, d\sigma$ equals the *Kaimanovich-entropy* h_K of the Brownian motion on $\tilde{M}$. In [19] Ledrappier showed:

Corollary. $\delta_0 \le \frac{1}{4}h_K$ *with equality if and only if M is asymptotically harmonic and hence locally symmetric.*

Proof. Using the constant function 1 in Theorem D we obtain $-\delta_0 \ge -\frac{1}{4}h_K$. Assume that the equality holds and let ϕ be a smooth function on T^1M with $\int \phi\, d\sigma = 0$. Then

$$\begin{aligned}\frac{d}{dt}\int(1+t\phi)[\Delta^s(t\phi) + Y(t\phi) - (1+t\phi)\frac{1}{4}\|Y\|^2]\, d\sigma\,|_{t=0} \\ = \int(\Delta^s(\phi) + Y(\phi) - \tfrac{1}{2}\phi\|Y\|^2)\, d\sigma = -\tfrac{1}{2}\int \phi\|Y\|^2\, d\sigma,\end{aligned}$$

since σ is a harmonic measure for $\Delta^s + Y$. But $t = 0$ is a maximum for the assignment

$$t \to \frac{\int(1+t\phi)[\Delta^s(t\phi) + Y(t\phi) - (1+t\phi)\frac{1}{4}\|Y\|^2]\, d\sigma}{\int(t^2\phi^2+1)\, d\sigma},$$

and hence the differentiation at $t = 0$ yields $0 = -\frac{1}{2}\int \phi\|Y\|^2\, d\sigma$. Since ϕ was arbitrarily chosen such that $\int \phi\, d\sigma = 0$, we conclude that $\|Y\|^2 \equiv h_K$.

Now write $Y = \langle X, Y\rangle X + Y^{ss}$ where Y^{ss} is a section of TW^{ss}. Let μ be the Bowen-Margulis measure on T^1M, i.e., the unique Φ^t-invariant Borel-probability measure whose entropy equals the topological entropy h of the geodesic flow. Since the pressure of the function $\langle X, Y\rangle$ vanishes [16] we have

$$h \le \int\langle X, Y\rangle\, d\mu \le (\int \mid \langle X, Y\rangle \mid^2\ d\mu)^{1/2} \le (\int \|Y\|^2\, d\mu)^{1/2} = h_K^{1/2}$$

with equality if and only if $Y^{ss} \equiv 0$. But $h_K \leq h^2$ [16], and hence $Y = \sqrt{h_K}X$. Thus $\operatorname{div}(X) \equiv -\sqrt{h_K}$ implying that the mean curvature of the horospheres in $\tilde{M}$ is constant, i.e., that M is asymptotically harmonic.

By the results of Benoist, Foulon, Labourie, Besson, Courtois, Gallot [7], [4], [5], the manifold M is therefore in fact locally symmetric.

Let now Z be the g^s-gradient of the *Lebesgue-Liouville measure* λ on T^1M. In the same way as above we obtain that $\delta_0 \leq \int \frac{1}{4}\|Z\|^2\, d\lambda$ with equality if and only if M is locally symmetric.

Let $P\colon T^1\tilde{M} \to \tilde{M}$ be the canonical projection. For every $x \in \tilde{M}$ the restriction π_x of the natural projection $\pi\colon T^1\tilde{M} \to \partial\tilde{M}$ to $T^1_x\tilde{M}$ is a homeomorphism. For $v \in T^1\tilde{M}$, denote moreover by θ_v the *Busemann function* at $\pi(v)$ which is normalized by $\theta_v(Pv) = 0$.

2. Harmonic Gromov - distances

For $\epsilon > 0$, again let $K_\epsilon\colon \tilde{M} \times \tilde{M} \times \partial\tilde{M} \to (0,\infty)$ be the Martin kernel of the operator $\Delta_\epsilon = \Delta + \delta_0 - \epsilon$. Recall that T^1M (resp. $T^1\tilde{M}$) admits a natural embedding into DTM (resp. $DT\tilde{M}$) by mapping $v \in T^1M$ (resp. $v \in T^1\tilde{M}$) to the element (v,v) of the diagonal in DTM (resp. $DT\tilde{M}$). With the notation from the introduction we then have:

Lemma 2.1. *For every $p \in \tilde{M}$ and $v \neq w \in T^1_p\tilde{M}$ the limit*

$$\beta_\epsilon(v,w) = \lim_{y \to \pi(v), z \to \pi(w)} \frac{1}{2}[\log G_\epsilon(z,y) - \log G_\epsilon(p,y) - \log G_\epsilon(z,p)]$$

exists. The function $\beta_\epsilon\colon DT\tilde{M} - T^1\tilde{M} \to \mathbb{R}$ is continuous and invariant under the action of $\pi_1(M)$ on $DT\tilde{M}$. Moreover for $(v,w),(z,u) \in DT\tilde{M}$ with $z \in W^s(v), u \in W^s(w)$ we have

$$\beta_\epsilon(v,w) - \beta_\epsilon(u,z) = \frac{1}{2}[\log K_\epsilon(Pv,Pu,\pi(v)) + \log K_\epsilon(Pv,Pu,\pi(w))].$$

Proof. By the Harnack inequality at infinity of Ancona and the arguments in the proof of Theorem 6.2 of Anderson-Schoen [3], for fixed $p, y \in \tilde{M}$ the function $z \to \frac{G_\epsilon(z,y)}{G_\epsilon(p,y)G_\epsilon(z,p)}$ has a Hölder continuous extension to the boundary, uniformly in $p, y \in \tilde{M}$. From this we conclude as in [17] that the limit $\beta_\epsilon(v,w)$ as above exists and depends continuously on $(v,w) \in DT\tilde{M}$. But also

$$\lim_{y \to \zeta}(\log G_\epsilon(p,y) - \log G_\epsilon(q,y)) = \log K_\epsilon(q,p,\zeta)$$

and from this we obtain the required formula for $\beta_\epsilon(v,w) - \beta_\epsilon(u,z)$.

Recall that we have a Hölder continuous foliation DW^s on $DT\tilde{M}$ and DTM with the property that the leaf $DW^s(v,w)$ of DW^s through a point $(v,w) \in DTM$ consists of all points $(u,z) \in DTM$ with $u \in W^s(v)$ and $z \in W^s(w)$. Then the first factor projection $R_1: DTM \to T^1M$ maps the foliation DW^s to the stable foliation. Moreover the natural embedding of T^1M into DTM is an embedding of the foliated space (T^1M, W^s) into the foliated space (DTM, DW^s).

Recall the definition of the *Gromov products* on $\partial\tilde{M}$ (see [9]); namely for $x \in \tilde{M}$ and $v \neq w \in T^1_x\tilde{M}$ define

$$(v|w) = \lim_{y \to \pi(v), z \to \pi(w)} \frac{1}{2}(\text{dist}(x,y) + \text{dist}(x,z) - \text{dist}(y,z)).$$

Clearly $(v|w) \geq 0$ for all $(v,w) \in DT\tilde{M}$, $(v|w) = 0$ if and only if $w = -v$, and for $(v,w) \in DT\tilde{M} - T^1\tilde{M}$ and $(u,z) \in DW^s(v,w)$ we have $(v|w) - (u|z) = \frac{-1}{2}(\theta_v(Pu) + \theta_w(Pu))$. Now the functions $(\,|\,)$ and β_ϵ on $DT\tilde{M} - T^1\tilde{M}$ are clearly invariant under the action of $\pi_1(M)$ on $DT\tilde{M} - T^1\tilde{M}$, and hence they project to functions on $DTM - T^1M$ which we denote by the same symbols. These functions can be compared as follows:

Lemma 2.2. *There is a number $\alpha > 0$ and for every $\epsilon \in (0, \delta_0]$ there is a number $c_\epsilon > 0$ such that $e^{-\alpha\beta_\epsilon(v,w)} \geq c_\epsilon e^{-(v|w)}$ for all $(v,w) \in DTM - T^1M$.*

Proof. Define $A = \{(v,w) \in DTM | \angle(v,-w) \leq \frac{\pi}{2}\}$. Then A is a compact subset of $DTM - T^1M$, and hence by continuity of the functions β_ϵ for fixed $\epsilon \in (0,\delta_0]$ there is a number $a_\epsilon > 0$ such that $\beta_\epsilon(v,w) \leq a_\epsilon$ for all $(v,w) \in A$.

Recall that the Riemannian metric on M can be lifted to a metric on the leaves of $DW^s \subset DTM$ in such a way that the norm of the leafwise gradient of the function $(\,|\,)$ with respect to this metric is bounded on $DTM - \{T^1M \cup A\}$ pointwise from below by a universal constant $b > 0$. Moreover by Lemma 2.1 and the Harnack inequalities the norm of the leafwise gradient of β_ϵ with respect to this metric is pointwise uniformly bounded on $DTM - T^1M$ by some constant $c > 0$ which is independent of $\epsilon \in (0, \delta_0]$. Let now $(v,w) \in DTM - \{A \cup T^1M\}$ and let $\phi: [0,\infty) \to DW^s(v,w)$ be the flow line of the gradient flow of the restriction of $-(\,|\,)$ to $DW^s(v,w)$. Then there is a minimal number $\tau > 0$ such that $\phi(\tau) \in A$ and we can estimate

$$(v|w) \geq \int_0^\tau \|\phi'(t)\|^2\,dt \geq b^2\tau.$$

On the other hand, in the same way we see that $\beta_\epsilon(v,w) \leq \beta_\epsilon(\phi(\tau)) + c\tau$. With $\alpha = b^2/c$ it follows that $\alpha\beta_\epsilon(v,w) \leq (v|w) + a_\epsilon\alpha$ for all $(v,w) \in$

$DTM - T^1M$. This shows the lemma.

Lemma 2.3. *For every* $\epsilon \in (0, \delta_0]$ *there are numbers* $\overline{\alpha}_\epsilon > 0, \overline{c}_\epsilon > 0$ *such that* $e^{-(v|w)} \geq \overline{c}_\epsilon e^{-\overline{\alpha}_\epsilon \beta_\epsilon(v,w)}$ *for all* $(v, w) \in DTM - T^1M$.

Proof. Fix again a number $\epsilon > 0$. The function $(\,|\,)$ on $DTM - T^1M$ assumes its minimum 0 precisely on the set $\{(v, -v) \mid v \in T^1M\}$. By compactness and continuity for fixed $\epsilon \in (0, \delta_0]$ there is further a number $a_\epsilon > 0$ such that $\beta_\epsilon(v, -v) \geq -a_\epsilon$ for all $v \in T^1M$.

Let now $(v, w) \in DT^1\tilde{M} - T^1\tilde{M}$ and identify the leaf $DW^s(v, w)$ of DW^s through (v, w) with $\tilde{M}$ via the projection $P \circ R^1$. Write $x = Pv$ and let A be the convex subset of $\tilde{M}$ of all points which lie on a geodesic joining $\pi(v)$ to $\pi(w)$. Denote by y the unique projection of x to A, let $\tau = \text{dist}(x, y) = \text{dist}(x, A)$ and let $z \in T^1_y\tilde{M}$ be such that $\pi(z) = \pi(v)$; then $x \in C(z, \frac{3}{4}\pi) \cap C(-z, \frac{3}{4}\pi)$, where for $u \in T^1\tilde{M}$ and $\gamma \in (0, \pi]$ we denote by $C(u, \gamma)$ the cone of angle γ and direction u in $\tilde{M}$.

Now the operator Δ_ϵ is coercive and hence its Green's function decays exponentially at infinity ([1]). Thus the Harnack inequality at infinity of Ancona together with continuity in v implies that there are numbers $b_\epsilon > 0, \alpha_\epsilon > 0$ such that $\frac{1}{2}(\log K_\epsilon(y, x, \pi(v)) + \log K_\epsilon(y, x, \pi(w))) \leq -\alpha_\epsilon\tau + b_\epsilon$.

This shows that $\beta_\epsilon(v, w) \geq \alpha_\epsilon\tau - a_\epsilon - b_\epsilon$. On the other hand, the norm of the gradient of $\frac{1}{2}(\theta_z + \theta_{-z})$ is bounded from above by 1 and consequently we obtain $(v|w) \leq \tau$. Thus $\beta_\epsilon(v, w) \geq \alpha_\epsilon(v|w) - a_\epsilon - b_\epsilon$ which implies the lemma.

Recall that $\tilde{M} \times \partial\tilde{M}$ is naturally homeomorphic to the unit tangent bundle $T^1\tilde{M}$ of $\tilde{M}$ by assigning the point $(Pv, \pi(v)) \in \tilde{M} \times \partial\tilde{M}$ to $v \in T^1\tilde{M}$. Thus for $\epsilon > 0$ there is a unique section $\tilde{\xi}_\epsilon$ of TW^s over $T^1\tilde{M}$ with the property that for every $v \in T^1\tilde{M}$ the restriction of $\tilde{\xi}_\epsilon$ to $W^s(v)$ projects to the gradient of the logarithm of the function $y \to K_\epsilon(Pv, y, \pi(v))$. As in Section 3 of [10] we deduce that $\tilde{\xi}_\epsilon$ is Hölder continuous. Moreover $\tilde{\xi}_\epsilon$ is clearly equivariant under the action of $\pi_1(M)$ and hence projects to a Hölder continuous section ξ_ϵ of TW^s over T^1M. In particular the assignment $v \to \langle X, \xi_\epsilon\rangle(v)$ is a Hölder continuous function on T^1M.

Let $\mathcal{M}$ be the space of Φ^t-invariant Borel-probability measures on T^1M. $\mathcal{M}$ is a compact convex subset of the dual of the Banach space $C^0(T^1M)$ of continuous functions on T^1M equipped with the weak*-topology. For $\eta \in \mathcal{M}$, denote by h_η the entropy of η as a Φ^t-invariant measure on T^1M. Recall that for a continuous function f on T^1M the *pressure* $pr(f)$ of f is defined by $pr(f) = \sup\{h_\eta - \int f d\eta \mid \eta \in \mathcal{M}\}$.

For $\epsilon > 0$ let $q(\epsilon)$ (resp. $r(\epsilon)$) be the pressure of the Hölder continuous function $2\langle X, \xi_\epsilon\rangle$ (resp. $\langle X, \xi_\epsilon\rangle$) on T^1M.

Lemma 2.4. *The assignments $\epsilon \to q(\epsilon)$ and $\epsilon \to r(\epsilon)$ are continuous and strictly decreasing on* $(0, \delta_0]$.

Proof. The considerations of Ancona [1] show that the assignment

$$T^1M \times (0, \delta_0] \to \mathbb{R}, (v, \epsilon) \to \langle X, \xi_\epsilon \rangle (v)$$

is continuous, and hence the function $q : \epsilon \in (0, \delta_0] \to q(\epsilon) \in \mathbb{R}$ is continuous as well (see [22]). To show that q is strictly decreasing for $v \in T^1\tilde{M}$ and $\epsilon > 0$, denote by u_v^ϵ the Δ_ϵ-harmonic function

$$y \in \tilde{M} \to u_v^\epsilon(y) = K_\epsilon(Pv, y, \pi(v))$$

with pole at $\pi(v)$. Let $\epsilon > \delta > 0$; the Harnack-inequality at infinity of Ancona [1] and his estimates for the Green's functions G_ϵ, G_δ of $\Delta_\epsilon, \Delta_\delta$ show that there is a number $c > 0$ depending on ϵ and δ but not on $v \in T^1\tilde{M}$ such that

$$\begin{aligned} cu_v^\epsilon(P\Phi^{-t}v) &\leq G_\epsilon(Pv, P\Phi^{-t}v) \leq c^{-1}e^{-ct}G_\delta(Pv, P\Phi^{-t}v) \\ &\leq c^{-2}e^{-ct}u_v^\delta(P\Phi^{-t}v) \end{aligned}$$

for all $t \geq 1$. If w is the projection of v to T^1M then

$$\begin{aligned} \log u_v^\epsilon(P\Phi^{-t}v) &= -\int_0^t \langle X, \xi_\epsilon \rangle (\Phi^{-s}w) ds \\ &\leq \log u_v^\delta(P\Phi^{-t}v) - ct - 3\log c \\ &= -\int_0^t \langle X, \xi_\delta \rangle (\Phi^{-s}w) ds - ct - 3\log c. \end{aligned}$$

Now let $\eta \in \mathcal{M}$ be ergodic with respect to Φ^t; by the Birkhoff ergodic theorem there is then $w \in T^1M$ such that

$$-\int \langle X, \xi_\epsilon \rangle d\eta = \lim_{t\to\infty} \frac{1}{t} \int_0^t \langle X, \xi_\epsilon \rangle (\Phi^{-s}w) ds$$

and

$$-\int \langle X, \xi_\delta \rangle d\eta = \lim_{t\to\infty} \frac{1}{t} \int_0^t \langle X, \xi_\delta \rangle (\Phi^{-s}w) ds$$

and consequently

$$-\int \langle X, \xi_\epsilon \rangle d\eta \leq -\int \langle X, \xi_\delta \rangle d\eta - c$$

by the above estimate. Since ergodic measures in $\mathcal{M}$ are just the extremal points of $\mathcal{M}$ this inequality then holds for every Φ^t-invariant Borel-probability measure η on T^1M. In other words we have

$$h_\eta - \int 2\langle X, \xi_\epsilon \rangle d\eta \le h_\eta - \int 2\langle X, \xi_\delta \rangle d\eta - 2c$$

for all $\eta \in \mathcal{M}$ and consequently $q(\epsilon) \le q(\delta) - 2c < q(\delta)$. The proof for $r(\epsilon)$ is completely analogous.

Recall from [12] and the introduction the definition of the g^s*-gradient* of a Borel measure ρ on T^1M which is absolutely continuous with respect to the stable and strong unstable foliation, with conditionals on stable manifolds in the Lebesgue measure class; namely, let $\tilde\rho$ be the lift of ρ to $T^1\tilde M$, and let $\tilde\rho(\infty)$ be a Borel-probability measure on $\partial\tilde M$ which defines the measure class of the projections of the conditionals of $\tilde\rho$ on strong unstable manifolds. For $v \in T^1\tilde M$ we can represent $\tilde\rho$ near v in the form $d\tilde\rho = \alpha d\lambda^s \times d\tilde\rho(\infty)$ where $\alpha : T^1\tilde M \to (0,\infty)$ is a Borel function, and we identify $\tilde\rho(\infty)$ with its projections to the leaves of W^{su} via the canonical projection $\pi : T^1\tilde M \to \partial\tilde M$.

For

$$(v,w) \in D = \{(u,z) \in T^1\tilde M \times T^1\tilde M \mid z \in W^s(u)\}$$

define $l(v,w) = \alpha(w)/\alpha(v)$. Then the function $l : D \to (0,\infty)$ is independent of the choice of $\tilde\rho(\infty)$. If for $\tilde\rho$-almost every $v \in T^1\tilde M$ the function $l_v : W^s(v) \to (0,\infty), w \to l_v(w) = l(v,w)$ is differentiable, then we obtain a measurable section $\tilde Z$ of TW^s over $T^1\tilde M$ by assigning to $v \in T^1\tilde M$ the gradient at v of $\log l_v$ with respect to the Riemannian metric g^s on $W^s(v)$. This section of TW^s over $T^1\tilde M$ is equivariant under the action of $\pi_1(M)$, and hence projects to a measurable section Z of TW^s over T^1M which we call the g^s-gradient of ρ. We then have $\int(\mathrm{div}(Y) + \langle Z, Y\rangle)d\rho = 0$ for every leafwise differentiable section Y of TW^s (see [12]) where for $v \in T^1M$ we denote by div $Y(v)$ the divergence at v of the restriction of Y to a vector field on $(W^s(v), \langle,\rangle) = (W^s(v), g^s)$.

Lemma 2.5. $q(\epsilon) < 0$ *for all* $\epsilon \in (0,\delta_0]$.

Proof. Ledrappier showed in [16] that the pressure of the function $\langle X, \xi_{\delta_0}\rangle$ vanishes; this implies $q(\delta_0) < 0$.

Assume to the contrary that $q(\tilde\epsilon) \ge 0$ for some $\tilde\epsilon > 0$. By continuity we then can find some $\epsilon \in (0,\delta_0]$ such that $q(\epsilon) = 0$.

Let ν^{su} be a family of conditional measures on strong unstable manifolds of the Gibbs equilibrium state ν_ϵ for the function $2\langle X, \xi_\epsilon\rangle$ with the property that $\frac{d}{dt}\nu^{su} \circ \Phi^t \mid_{t=0} = 2\langle X, \xi_\epsilon\rangle$. Let ν be the finite Borel measure on T^1M which satisfies $d\nu = d\lambda^s \times d\nu^{su}$; then the g^s-gradient of ν equals $2\xi_\epsilon$.

Let $\delta \in (0,\epsilon)$; then div $\xi_\delta + \parallel \xi_\delta \parallel^2 + \delta_0 - \delta = 0$ and consequently

$$\begin{aligned} 0 &= \int (\operatorname{div}(\xi_\delta - \xi_\epsilon) + 2\langle \xi_\epsilon, \xi_\delta - \xi_\epsilon\rangle) d\nu \\ &= \int (- \parallel \xi_\delta \parallel^2 + \delta - \epsilon - \parallel \xi_\epsilon \parallel^2 + 2\langle \xi_\epsilon, \xi_\delta \rangle) d\nu \\ &= \int (- \parallel \xi_\delta - \xi_\epsilon \parallel^2 + \delta - \epsilon) d\nu, \end{aligned}$$

which is possible only if $\delta \geq \epsilon$. From this we derive a contradiction to our assumption $q(\epsilon) = 0$.

Corollary 2.6. *For every $\epsilon \in (0, \delta_0]$ there is a unique number $a(\epsilon) \in [1,2)$ such that $pr(a(\epsilon)\langle X, \xi_\epsilon\rangle) = 0$, and moreover $a(\delta_0) = 1$.*

Proof. The fact that $pr(\langle X, \xi_{\delta_0}\rangle) = 0$ follows from the results of Ledrappier [16]. Let $\epsilon \in (0, \delta_0)$; then $r(\epsilon) > 0$ and $q(\epsilon) < 0$ by Lemma 2.4 and Lemma 2.5. On the other hand, the function $s \to pr(s\langle X, \xi_\epsilon\rangle)$ is continuous and hence has to vanish for some $a(\epsilon) \in (1,2)$. This number $a(\epsilon)$ is unique (a fact that is not needed in the sequel).

For $\epsilon > 0$ let ω_ϵ be the unique Gibbs-equilibrium state of the function $a(\epsilon)\langle X, \xi_\epsilon\rangle$. Then ω_ϵ admits a family ω_ϵ^{su} of conditional measures on strong unstable manifolds with the following properties:

1) The measures ω_ϵ^{su} are locally finite, positive on open sets and absolutely continuous with respect to the stable foliation.
2) The measure $\overline{\omega}_\epsilon$ on T^1M which is defined by $d\overline{\omega}_\epsilon = d\lambda^s \times d\omega_\epsilon^{su}$ has total mass 1 and its g^s-gradient equals $a(\epsilon)\xi_\epsilon$.

For every $x \in \tilde{M}$ the projection $\pi\colon T^1\tilde{M} \to \partial\tilde{M}$ restricts to a homeomorphism π_x of $T_x^1\tilde{M}$ onto $\partial\tilde{M}$, and for every $v \in T_x^1\tilde{M}$ the restriction of $\pi_x^{-1} \circ \pi$ to $W^{su}(v)$ is a homeomorphism of $W^{su}(v)$ onto $T_x^1\tilde{M} - \{-v\}$. Thus the measure $\tilde{\omega}_\epsilon^{su}$ on $W^{su}(v)$ which is lifted from the measures ω_ϵ^{su} on the leaves of $W^{su} \subset T^1M$ projects under $\pi_x^{-1} \circ \pi|_{W^{su}(v)}$ to a Borel-measure ω_ϵ^v on $T_x^1\tilde{M}$, whose restriction to $T_x^1\tilde{M} - \{-v\}$ is locally finite. The measures $\omega_\epsilon^v, \omega_\epsilon^w (v, w \in T_x^1\tilde{M})$ are absolutely continuous on $T_x^1\tilde{M} - \{-v, -w\}$, with continuous Radon-Nikodym-derivative. More precisely, for $w \in T_x^1\tilde{M} - \{-v\}$ the Radon-Nikodym-derivative $J_v^\epsilon(w)$ at ω of ω_ϵ^w with respect to ω_ϵ^v is defined and the function $J_v^\epsilon\colon w \to J_v^\epsilon(w)$ is continuous on $T_x^1\tilde{M} - \{-v\}$. Thus we obtain a Borel-measure ω_ϵ^x on $T_x^1\tilde{M}$ by defining $\omega_\epsilon^x = J_v^\epsilon \omega_\epsilon^v$. Since $\omega_\epsilon^x = J_w^\epsilon \omega_\epsilon^w$ for every $w \in T_x^1\tilde{M}$, the measure ω_ϵ^x is defined independent of the choice of $v \in T^1\tilde{M}$ and is finite.

For $v \in T^1\tilde{M}$ and $t > 0$ the homeomorphism $\pi_{P\Phi^t v}^{-1} \circ \pi_{Pv}\colon T_{Pv}^1\tilde{M} \to T_{P\Phi^t v}^1\tilde{M}$ is absolutely continuous with respect to the measures $\omega_\epsilon^{Pv}, \omega_\epsilon^{P\Phi^t v}$, and its Jacobian at v equals $e^{a(\epsilon)\int_0^t \langle X, \tilde{\xi}_\epsilon\rangle(\Phi^s v)\, ds}$. Moreover the measures

ω_ϵ^x $(x \in \tilde{M})$ are equivariant under the action of the fundamental group $\pi_1(M)$ of M on $T^1\tilde{M}$, and hence induce for every $p \in M$ a finite measure ω_ϵ^p on $T_p^1 M$. The measures $\omega_{\delta_0}^p (p \in M)$ just coincide with the harmonic measures ω^p from the introduction up to a universal constant.

Let $\rho > 0$. Following Margulis [20] we call two subsets B_1, B_2 of T^1M which are contained in leaves $T_x^1 M, T_y^1 M$ of the *vertical foliation* of T^1M into the fibres of the fibration $T^1M \to M$ *ρ-equivalent* if there is a continuous map $f : B_1 \times [0,1] \to T^1M$ with the following properties:

i) For every $v \in B_1$ the set $f(\{v\} \times [0,1])$ is a smooth curve of length smaller than ρ in $W^s(v)$.
ii) $f(v,0) = v$ and $f(v,1) \in B_2$ for all $v \in B_1$.
iii) The map $v \in B_1 \to f(v,1) \in B_2$ is a homeomorphism.

With this notation we then have:

Lemma 2.7. *For every $\delta > 0$ there is a number $\rho = \rho(\delta) > 0$ such that*

$$\omega_\epsilon^p(A)/\omega_\epsilon^q(B) < \delta + 1$$

for all $\epsilon \in (0, \delta_0)$ and all ρ-equivalent nontrivial open subsets A, B of leaves of the vertical foliation. In particular, there is for every $\gamma > 0$ a number $c = c(\gamma) > 0$ such that

$$\omega_\epsilon^{Pv}\{w \in T_{Pv}^1 M \mid \angle(v,w) < \gamma\} \in [c^{-1}, c]$$

for all $v \in T^1M$ and all $\epsilon \in (0, \delta_0]$.

Proof. Let $C \subset T^1M$ be a set with a local product structure, given by a vector $v \in T^1M$, a number $r > 0$, the open ball $B^s(v,r)$ of radius r about v in $(W^s(v), \langle,\rangle)$, the open ball $B^v(v,r) = \{w \in T_{Pv}^1 M \mid \angle(v,w) < r\}$ of radius r about v in $T_{Pv}^1 M$ with respect to the angular metric and a homeomorphism $[\ ,\ ] : B^s(v,r) \times B^v(v,r) \to C$ with the following properties:

i) $[w,v] = w$ for all $w \in B^s(v,r)$.
ii) $[v,z] = z$ for all $z \in B^v(v,r)$.
iii) $[w,z] \in W^s(z) \cap T_{Pw}^1 M$ for all $w \in B^s(v,r)$, all $z \in B^v(v,r)$.

Let $\epsilon > 0$; then for every $z \in B^s(v,r)$ the canonical map which assigns to $w \in B^v(v,r)$ the point $[z,w] \in T_{Pz}^1 M$ is absolutely continuous with respect to the measures ω_ϵ^p, and its Jacobian $J(z,w)$ at w equals the value at z of the unique function ϕ_w on $[B^s(v,r), w]$ which satisfies $\phi_w(w) = 1$ and whose gradient with respect to the metric $\langle,\rangle$ on $W^s(w) \supset [B^s(v,r), w]$ equals $a(\epsilon)\xi_\epsilon$. Since by the Harnack inequality for positive Δ_ϵ-harmonic functions the vector fields ξ_ϵ are pointwise uniformly bounded in norm, independent of $\epsilon \in (0, \delta_0]$, the first part of the lemma follows from the definition of ρ-equivalence.

Choose now $r > 0$ sufficiently small that for every $v \in T^1M$ there is a subset of T^1M with a local product structure containing $B^v(v,r)$ and $B^s(v,r)$. Define a finite Borel measure $\overline{\omega}_\epsilon$ on T^1M by $d\overline{\omega}_\epsilon(v) = d\lambda^s \times d\omega_\epsilon^{Pv}(v)$ (in fact this measure coincides with the Borel probability measure- equally denoted by $\overline{\omega}_\epsilon$- which was defined after Corollary 2.6, see [14]). Thus there is a number $a > 0$ such that

$$\begin{aligned} a^{-1}\lambda^s(B^s(v,r))\omega_\epsilon^{Pv}(B^v(v,r)) &\le \bar{\omega}_\epsilon[B^s(v,r),B^v(v,r)] \\ &\le a\lambda^s(B^s(v,r))\omega_\epsilon^{Pv}(B^v(v,r)) \end{aligned}$$

for all $v \in T^1M$ and all $\epsilon > 0$. Since by the definition of λ^s there is a number $b > 0$ such that $\lambda^s(B^s(v,r)) \in [b^{-1},b]$ for all $v \in T^1M$ and moreover $0 < \overline{\omega}_\epsilon(T^1M) < \infty$, we obtain the existence of a number $C_0 > 0$ not depending on $\epsilon \in (0,\delta_0]$ such that $\omega_\epsilon^{Pv}(B^v(v,r)) \le C_0$ for all $v \in T^1M$.

Now let $\tilde{\omega}_\epsilon$ be the lift of $\overline{\omega}_\epsilon$ to $T^1\tilde{M}$. Since every leaf of W^s is dense in T^1M, there is a number $R > 0$ such that for every $\tilde{v} \in T^1\tilde{M}$ the subset $\tilde{C}$ of $T^1\tilde{M}$ with a local product structure which is defined by $\tilde{C} \cap W^s(\tilde{v}) = B^s(\tilde{v},R)$ and $\tilde{C} \cap T^1_{Pv}\tilde{M} = B^v(v,r)$ projects onto T^1M. The above arguments applied to $\tilde{\omega}_\epsilon$ then show $\tilde{\omega}_\epsilon(\tilde{C}) \le$ const. $\omega_\epsilon^{P\tilde{v}}B^v(\tilde{v},r)$ where the constant does not depend on $\tilde{v}$ and ϵ. But $\tilde{\omega}_\epsilon(\tilde{C}) \ge$ const. and this implies that the measures $\omega_\epsilon^{Pv}(B^v(v,r))$ are bounded from below by a universal constant as well. These arguments are valid for all sufficiently small $r > 0$ and from this the lemma follows.

For $\epsilon \in (0,\delta_0]$ let again β_ϵ: $DT\tilde{M} - T^1\tilde{M} \to [0,\infty)$ and $a(\epsilon) \in [1,2)$ be as before. For $v \in T^1\tilde{M}$ and $\rho > 0$ let

$$B_\epsilon(v,\rho) = \{w \in T^1_{Pv}\tilde{M} | e^{-\beta_\epsilon(v,w)} \le \rho\};$$

this is a closed neighborhood of v in $T^1_{Pv}\tilde{M}$. For $p \in \tilde{M}$ and a Borel-subset A of $T^1_p\tilde{M}$ write

$$\zeta_\epsilon^p(A) = \sup_{i>0} \inf\Big\{ \sum_{j=1}^\infty \rho_j^{a(\epsilon)} \mid \rho_j \le 1/i \ (j \ge 1)$$

$$\text{and} \quad A \subset \cup_{j=1}^\infty B_\epsilon(v_j,\rho_j) \quad \text{for some} \quad v_j \in T^1_p\tilde{M}\Big\}.$$

Then ζ_ϵ^p is a Borel-measure on $T^1_p\tilde{M}$ (which a priori might be zero or infinite). Moreover the measures ζ_ϵ^p project to families of Borel measures on the fibres of $T^1M \to M$ which we denote by the same symbols.

Now we obtain the following generalization of Theorem A from the introduction:

Proposition 2.8. *For every $\epsilon > 0$ there is a number $b_\epsilon > 0$ such that $\zeta_\epsilon^p = b_\epsilon \omega_\epsilon^p$ for all $p \in \tilde{M}$.*

Proof. We show first that the measures ζ_ϵ^p are finite, and define the same measure class as the measures ω_ϵ^p $(p \in \tilde{M})$. For this let $c > 0$ be such that for every $v \in T^1\tilde{M}$, every $t \geq 0$ and every $w \in T^1_{Pv}\tilde{M}$ with $\angle(v,w) < \pi/4$ we have

$$K_\epsilon(Pv, P\Phi^{-t}v, \pi(v))/K_\epsilon(Pv, P\Phi^{-t}v, \pi(w)) \in [c^{-1}, c];$$

such a number exists by the Harnack inequality at infinity of Ancona.

Fix a number $r > 0$ which is small enough that for every $v \in T^1\tilde{M}$ we have $B_\epsilon(v,r) \subset \{w \in T^1_{Pv}\tilde{M} | \angle(v,w) < \frac{\pi}{4}\}$; such a number exists by Lemma 2.2. By Lemma 2.3 there is then a number $\alpha > 0$ such that $B_\epsilon(v, c^{-1}r) \supset \{w \in T^1_{Pv}\tilde{M} | \angle(v,w) \leq \alpha\}$ for all $v \in T^1\tilde{M}$, and consequently Lemma 2.7 shows that $\omega_\epsilon^p(B_\epsilon(v,c^{-1}r)) \geq \kappa > 0$ for all $p \in \tilde{M}, v \in T^1_p\tilde{M}$ where κ is a universal constant.

Let $p \in \tilde{M}, v \in T^1_p\tilde{M}$ and let $\rho \leq c^{-1}r$. By continuity there is a number $\tau > 0$ such that $K_\epsilon(Pv, P\Phi^\tau v, \pi(v))\rho = r$. For $w \in B_\epsilon(\Phi^\tau v, c^{-1}r)$ and $u = \pi_p^{-1}(\pi(w))$ we then have

$$\begin{aligned} e^{-\beta_\epsilon(v,u)} &= K_\epsilon(Pv, P\Phi^\tau v, \pi(v))^{-1/2} K_\epsilon(Pv, P\Phi^\tau v, \pi(w))^{-1/2} e^{-\beta_\epsilon(w,\Phi^\tau v)} \\ &\leq K_\epsilon(Pv, P\Phi^\tau v, \pi(v))^{-1} r = \rho, \end{aligned}$$

and consequently $\pi_p(B_\epsilon(\Phi^\tau v, c^{-1}r)) \subset B_\epsilon(v,\rho)$. Lemma 3.6 of [10] and the Harnack inequality at infinity of Ancona thus imply that there is a number $\chi > 0$ such that

$$\omega_\epsilon^p(B(v,\rho)) \geq K_\epsilon(Pv, P\Phi^t v, \pi(v))^{-a(\epsilon)} r^{a(\epsilon)} \chi = \chi\rho^{a(\epsilon)}.$$

On the other hand, choose $s > 0$ such that $K_\epsilon(Pv, P\Phi^s v, \pi(v))\rho = c^{-1}r$. Let $w \in T_{P\Phi^s v}\tilde{M}$ with $e^{-\beta_\epsilon(\Phi^s v, w)} = r$ and let $u = \pi_p(w)$. Then

$$e^{-\beta_\epsilon(v,u)} \geq c^{-1} K_\epsilon(Pv, P\Phi^s v, \pi(v))^{-1} r = \rho,$$

and consequently $B_\epsilon(v,\rho) \subset \pi_p B_\epsilon(\Phi^s v, r)$. As before this means that there is $\bar{\chi} > 0$ such that $\omega_\epsilon^p(B(v,\rho)) \leq \bar{\chi}\rho^{a(\epsilon)}$. In other words, for every $v \in T^1\tilde{M}$ and every $\rho \leq r$ we have $\chi\rho^{a(\epsilon)} \leq \omega_\epsilon^p(B(v,\rho)) \leq \bar{\chi}\rho^{a(\epsilon)}$. This implies in particular that $\zeta_\epsilon^p \geq \bar{\chi}^{-1}\omega_\epsilon^p$ for all $p \in \tilde{M}$.

Let $\kappa > 0$ be sufficiently small that $e^{-\kappa\beta_\epsilon}$ satisfies the quasi-ultrametric inequality [14] on the fibres $T^1_p\tilde{M}$ $(p \in \tilde{M})$; such a number exists by Lemma 2.2 and Lemma 2.3. Let $\rho > 0$ and let $v_1, \dots, v_{k(\rho)} \in T^1_p\tilde{M}$ be a maximal system of points such that the balls $B_\epsilon(v_i, \rho) \subset T^1_p\tilde{M}$ are pairwise disjoint. Then the balls $B_\epsilon(v_i, 4^{1/\kappa}\rho)$ cover $T_p^1\tilde{M}$ and hence

$$\zeta_\epsilon^p(T_p^1\tilde{M}) \leq \limsup_{\rho\to 0} k(\rho)\cdot 4^{1/\kappa}\rho^{a(\epsilon)}$$
$$\leq 4^{1/\kappa}\chi^{-1}\limsup_{\rho\to 0}\omega_\epsilon^p(\cup_{i=1}^{k(\rho)}B_\epsilon(v_i,\rho)) \leq 4^{1/\kappa}\chi^{-1}.$$

In other words, the measures ζ_ϵ^p $(p\in\tilde{M})$ are finite and define the same measure class as the measures ω_ϵ^p.

We are left with showing that $\zeta_\epsilon^p = b_\epsilon\omega_\epsilon^p$ with a universal constant $b_\epsilon > 0$. Since by their definition the measures ζ_ϵ^p are equivariant under the action of $\pi_1(M)$ it suffices for this to prove that for $p\in\tilde{M}, v\in T_p^1\tilde{M}$ and $t\in\mathbb{R}$ the Jacobian of the projection π_p with respect to the measures $\zeta_\epsilon^{P\Phi^t v}$ and ζ_ϵ^p at $\Phi^t v$ equals $K_\epsilon(P\Phi^t v, Pv, \pi(v))^{a(\epsilon)}$. But this is a direct consequence of the definitions and the fact that

$$\lim_{w\to\Phi^t v} e^{-\beta_\epsilon(w,\Phi^t v)}/e^{-\beta_\epsilon(\pi_p(w),v)} = K(P\Phi^t v, Pv, \pi(v)).$$

3. Asymptotic properties of the Green's function for $\Delta + \delta_0$

This section is devoted to the proof of the first part of Theorem B in the introduction. We resume the assumptions and notation of Sections 1 and 2. In particular recall the definition of the Hölder-continuous sections $\langle X, \xi_\epsilon\rangle$ of TW^s over T^1M for $\epsilon > 0$.

First we estimate for $a\in[1,4]$ and $\epsilon\in(0,\delta_0]$ the entropy of the unique Gibbs equilibrium state for the function $a\langle X,\xi_\epsilon\rangle$.

Lemma 3.1. *There is a number $\chi > 0$ such that for every $a\in[1,4]$ and every $\epsilon\in(0,\delta_0]$ the entropy of the unique Gibbs equilibrium state for the function $a\langle X,\xi_\epsilon\rangle$ is not smaller than χ.*

Proof. By the Harnack-inequality the functions $a\langle X,\xi_\epsilon\rangle$ are pointwise uniformly bounded in norm, independent of $a\in[1,4]$ and $\epsilon\in(0,\delta_\epsilon]$. Thus if we define $p(a,\epsilon)$ to be the pressure of the function $a\langle X,\xi_\epsilon\rangle$, then this defines a continuous function $p\colon [1,4]\times(0,\delta_0]\to\mathbb{R}$ which is uniformly bounded by a number $\rho > 0$.

Identify the diagonal $\{(v,v)\in DTM \mid v\in T^1M\}$ of DTM with T^1M. For $(v,w)\in DTM - T^1M$, again let $(v|w)$ be the Gromov-product of v and w, and for $(a,\epsilon)\in[1,4]\times(0,\delta]$ and $(v,w)\in DTM - T^1M$ define $\delta(a,\epsilon)(v,w) = e^{-a\beta_\epsilon(v,w)-p(a,\epsilon)(v|w)}$. The function $\delta(a,\epsilon)$ is continuous, symmetric and admits a continuous extension by zero to the diagonal.

We claim that there is a number $b > 0$ and for every $(a,\epsilon)\in[1,4]\times(0,\delta_0]$ a number $c(a,\epsilon) > 0$ such that $\delta(a,\epsilon)(v,w)\geq c(a,\epsilon)e^{-b(v|w)}$ for all

$(v,w) \in DTM$. For this simply recall from Lemma 2.2 that $e^{-\beta_\epsilon(v,w)} \geq c_\epsilon e^{-(v|w)/\alpha}$ for all $\epsilon \in (0,\delta_0]$ and all $(v,w) \in DTM$, where $\alpha > 0$ is a universal constant and $c_\epsilon > 0$ depends on ϵ.

For $p \in M$ let now $\nu(a,\epsilon)^p$ be the measure on T^1_pM obtained as in Section 2 from the conditionals of the Gibbs-equilibrium state $\nu(a,\epsilon)$ for $a\langle X, \xi_\epsilon\rangle$, and let μ^p be the measure induced from the conditionals of the Bowen-Margulis measure. The arguments in the proof of Proposition 2.8 then show that up to a universal constant the measure $\nu(a,\epsilon)^p$ is just the 1-dimensinal spherical measure induced by the "distance" $\delta(a,\epsilon)$ on T^1_pM, while μ^p is up to a universal constant the h-dimensional spherical measure induced by the "distance"

$$\rho\colon (v,w) \to e^{-(v|w)},$$

where $h > 0$ is the topological entropy of the geodesic flow on T^1M. Since $\delta(a,\epsilon) \geq c(a,\epsilon)\rho^b$ this means that the Hausdorff dimension of the measure $\nu(a,\epsilon)^p$ with respect to the "distance" ρ on T^1_pM is not smaller than $1/b$. On the other hand, by [11] this Hausdorff dimension (which is independent of $p \in M$) is just the entropy of the Gibbs-measure $\nu(a,\epsilon)$. This shows the lemma.

Corollary 3.2. *For every $\epsilon > 0$ the pressure of the function $4\langle X, \xi_\epsilon\rangle$ is not larger than $-\chi$, where $\chi > 0$ is as in Lemma 3.1.*

Proof. Let $\epsilon > 0$ and let ν be the unique Gibbs-equilibrium state of the function $4\langle X, \xi_\epsilon\rangle$; then $h_\nu \geq \chi$ by Lemma 3.1. On the other hand, by Lemma 2.5 the pressure of the function $2\langle X, \xi_\epsilon\rangle$ is non-positive and consequently $0 \geq h_\nu - 2\int\langle X, \xi_\epsilon\rangle\, d\nu \geq \chi - 2\int\langle X, \xi_\epsilon\rangle\, d\nu$. From this we conclude that

$$h_\nu - 4\int\langle X, \xi_\epsilon\rangle\, d\nu = pr(4\langle X, \xi_\epsilon\rangle) \leq h_\nu - 2\int\langle X, \xi_\epsilon\rangle\, d\nu - \chi \leq -\chi$$

which shows the corollary.

Corollary 3.3. *$\int\langle X, \xi_\epsilon\rangle\, d\eta \geq \chi/4$ for every $\eta \in \mathcal{M}$ and every $\epsilon \in (0,\delta_0]$.*

Proof. Let η be a Φ^t-invariant Borel-probability measure on T^1M. Then $h_\eta \geq 0$ and $h_\eta - 4\int\langle X, \xi_\epsilon\rangle\, d\eta \leq -\chi$ by Corollary 3.2 from which the corollary follows.

Corollary 3.4. *The operator $\Delta + \delta_0$ admits a Green's function G_0, and the $\Delta + \delta_0$ - Martin boundary does not consist of a single point.*

Proof. Let $\gamma\colon \mathbb{R} \to \tilde{M}$ be a geodesic in $\tilde{M}$ whose projection to M is closed of length $\tau > 0$. For $\epsilon > 0$, denote by f^+_ϵ the unique minimal positive Δ_ϵ-harmonic function on $\tilde{M}$ with pole at $\gamma(\infty)$ which is normalized by $f^+_\epsilon(\gamma(0)) = 1$. Let $w \in T^1M$ be the projection of $\gamma'(0) \in T^1\tilde{M}$. Then w is a periodic point for Φ^t of period $\tau > 0$, and

$f_\epsilon(\gamma(\tau)) = e^{\int_0^\tau \langle X,\xi_\epsilon\rangle(\Phi^s w)\,ds} \geq e^{\tau\chi/4} > 1$ by Corollary 3.3. Since the space of positive Δ_ϵ-harmonic functions ($\epsilon \in (0,\delta]$) on $\tilde{M}$ which are normalized at $\gamma(0)$ is precompact with respect to uniform convergence on compact sets, we can find a sequence $\{\epsilon_j\} \subset (0,\delta_0]$ such that $\epsilon_j \to 0 \quad (j \to \infty)$ and that the functions $f^+_{\epsilon_j}$ converge uniformly on compact subsets of $\tilde{M}$ to a Δ_0-harmonic function f_0^+. Clearly $f_0^+(\gamma(\tau))/f_0^+(\gamma(0)) \geq e^{\tau\chi/4} > 1$.

On the other hand, the same argument applied to the geodesic $t \to \gamma\ (-t+\tau)$ whose tangent projects to the periodic orbit of Φ^t through $-w$, yields a positive Δ_0-harmonic function f_0^- on $\tilde{M}$ which satisfies

$$f_0^-(\gamma(\tau))/f_0^-(\gamma(0)) \leq e^{-\tau\chi/4} < 1.$$

But this means that f_0^- and f_0^+ are not constant multiples of each other. By the results of Sullivan [21] we conclude from this that Δ_0 admits a Green's function and further that the Δ_0-Martin boundary of $\tilde{M}$ does not consist of a single point.

Write now $p(\epsilon) = pr(4\langle X,\xi_\epsilon\rangle)$ and let η_ϵ be the Gibbs equilibrium state of the function $4\langle X,\xi_\epsilon\rangle$. Then η_ϵ admits a unique family η_ϵ^{su} of conditional measures on strong unstable manifolds which transform under the geodesic flow via $\frac{d}{dt}\{\eta_\epsilon^{su}\circ\Phi^t\}|_{t=0} = 4\langle\xi_\epsilon,X\rangle - p(\epsilon)$ and such that the measure $\overline{\eta}_\epsilon$ on T^1M which is defined by $d\overline{\eta}_\epsilon = d\lambda^s \times d\eta_\epsilon^{su}$ has total mass 1.

We use these measures to define as in Section 2 a family of finite Borel-measures η_ϵ^p ($p \in M$) on the leaves of the vertical foliation of T^1M. As in Section 2 we arrive at

Lemma 3.5. *For every $\delta > 0$ there is a number $\rho = \rho(\delta) > 0$ such that*

$$\eta_\epsilon^p(A)/\eta_\epsilon^q(B) < \delta + 1$$

for all $\epsilon > 0$ and all ρ-equivalent nontrivial open subsets A, B of leaves of the vertical foliation. In particular, there is a number $c > 0$ such that $\eta_\epsilon^p(T_p^1M) \in [c^{-1}, c]$ for all $p \in T^1M$ and all $\epsilon > 0$.

For $p \in \tilde{M}$ and $R > 0$ let $S(p,R)$ be the distance sphere of radius R about p in $\tilde{M}$ and let $\lambda_{p,R}$ be the Lebesgue measure on $S(p,R)$. Write

$$p(0) = \lim_{\epsilon\to 0} p(\epsilon) \leq -\chi.$$

Corollary 3.6. *There is a number $\tilde{c} > 0$ such that*

$$\int_{S(p,R)} G_\epsilon(p,y)^4 e^{-p(\epsilon)R}\, d\lambda_{p,R} \leq \tilde{c}$$

for all $p \in \tilde{M}$, all $R \geq 1$ and all $\epsilon \in [0,\delta_0]$.

Proof. By the maximum principle for positive Δ_ϵ-harmonic functions on $\tilde{M}$ ($\epsilon \in [0,\delta_0]$) there is a number $a > 0$ not depending on ϵ such that for all $p, x \in \tilde{M}$ with $\operatorname{dist}(p,x) \geq 1$ and every positive Δ_ϵ-harmonic function f on $\tilde{M}$ with $f(p)=1$ we have $G_\epsilon(p,x) \leq a^{-1} f(x)$.

For $w \in T^1\tilde{M}$ the Jacobian $J_\epsilon(w,t)$ of Φ^{-t} at $\Phi^t w$ with respect to the measures η_ϵ^p on the leaves of the vertical foliation equals

$$K_\epsilon(P\Phi^t w, Pw, \pi(w))^4 e^{-p(\epsilon)t} \geq a G_\epsilon(Pw, P\Phi^t w)^4 e^{-p(\epsilon)t} \quad (t \geq 1),$$

and hence Lemma 3.5 together with the Harnack inequalities shows that there is a constant $b > 0$ not depending on $\epsilon \in [0, \delta_0], w \in T^1\tilde{M}$ and $t \geq 1$ such that for every $v \in T^1\tilde{M}$ and every $t \geq 1$ we have

$$\eta_\epsilon^{Pv}\{w \in T^1_{Pv}\tilde{M} \mid \operatorname{dist}(P\Phi^t w, P\Phi^t v) \leq 1\} \geq b e^{-p(\epsilon)t} G_\epsilon(Pv, P\Phi^t v)^4.$$

Since the total mass $\eta_\epsilon^p(T_p^1\tilde{M})$ of $T_p^1\tilde{M}$ with respect to η_ϵ^p is bounded from above by a positive constant not depending on $\epsilon \in [0,\delta_0]$ and $p \in \tilde{M}$, a further application of the Harnack inequality for the Green's function yields the corollary (compare the proof of Corollary 3.13 in [10]).

Now we are ready for the proof the first part of Theorem B:

Corollary 3.7. *There is a number $c > 0$ such that $G_0(x,y) \leq c e^{-\chi \operatorname{dist}(x,y)/4}$ for all $x, y \in \tilde{M}$ with* $\operatorname{dist}(x,y) \geq 1$.

Proof. Since $p(0) \leq -\chi$, Corollary 3.6 implies that the integrals $\int_{S(x,R)} G_0^4(x,y) e^{\chi R}\, d\lambda_{x,R}(y)$ are bounded from above by a constant $a > 0$ which is independent of $x \in \tilde{M}$ and $R \geq 1$. Let $R_0 \geq 1$ be sufficiently large that $\lambda_{x,R} S(x,R) \geq 1$ for every $x \in \tilde{M}$ and $R \geq R_0$.

The Harnack-inequality for positive Δ_0-harmonic functions on balls shows that for $x, y \in \tilde{M}$ with $R = \operatorname{dist}(x,y) \geq R_0$, there is a ball B about y in $S(x,R)$ with $\lambda_{x,R}(B) = 1$ and such that $G_0(x,z) \geq \rho G_0(x,y)$ for all $z \in B$, where $\rho > 0$ is a universal constant. Now if $G_0(x,y) \geq 2a^{1/4}\rho^{-1/4} e^{-\chi \operatorname{dist}(x,y)/4}$, then this implies $\int_B G_0^4(x,y) e^{\chi \operatorname{dist}(x,y)}\, d\lambda_{x,R} \geq 8a$, a contradiction to the above.

4. A variational equation for δ_0

The purpose of this section is to prove Theorem D. For this let η as in the introduction be a Borel-probability measure on T^1M which can be written with respect to a local product structure in the form $d\eta = d\lambda^s \times d\eta^{su}$, where η^{su} is a family of locally finite Borel measures on the leaves of the strong unstable foliation, such that the g^s-gradient Y of η is of class $C_s^{1,\alpha}$. Since $\langle X, Y\rangle = \frac{d}{dt}\eta^{su} \circ \Phi^t \mid_{t=0}$, the family η^{su} is in fact a family of conditional measures on strong unstable manifolds of the unique Gibbs equilibrium state induced by the Hölder continuous function $\langle X, Y\rangle$. In other words, there is a family η^{ss} of conditional measures on strong stable manifolds such that the Borel-probability measure $\bar{\eta}$ on T^1M, which is defined with respect to a local product structure by $d\bar{\eta} = d\eta^{ss} \times d\eta^{su} \times dt$, is invariant under the geodesic flow.

For $v \in T^1M$, and $t \in \mathbb{R}$, define $\zeta(v,t) = \zeta_t(v) = e^{\int_0^t \langle X,Y\rangle(\Phi^s v)\, ds}$; then ζ is a multiplicative cocyle with respect to the geodesic flow.

Let $v \in T^1M$ and let $A \subset W^{ss}(v)$ be a compact ball with nonempty interior whose boundary is a set of measure zero with respect to η^{ss}. Denote by λ^{ss} the Lebesgue measure on the leaves of W^{ss} defined by the lift of the Riemannian metric on M. For every $t \in \mathbb{R}$ we then can view the restriction of λ^{ss} to $\Phi^t A$ as a finite Borel measure on T^1M. The arguments of Ledrappier in [17] then imply the following:

Proposition 4.1. *The measures $(\zeta_{-t} \circ \Phi^t)\lambda^{ss}\,|_{\Phi^{-t}A}$ converge as $t \to \infty$ weakly to the measure $\eta^{ss}(A)\eta$.*

This is used to show:

Lemma 4.2. *Let*

$$\alpha_\eta = \sup\{\int \phi(\Delta^s(\phi) + Y(\phi) + \phi[\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2])\, d\eta \mid$$
$$0 \not\equiv \phi \in C^\infty(T^1M), \int \phi^2\, d\eta = 1\};$$

then $-\delta_0 \geq \alpha_\eta$.

Proof. Define α_η as in the statement of the lemma; we show first that $\alpha_\eta < \infty$. For this recall that the function

$$v \to (\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2)(v)$$

is continuous and hence bounded on T^1M, and consequently

$$\int \phi^2[\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2]\, d\eta / \int \phi^2\, d\eta$$

is uniformly bounded for all nontrivial continuous functions ϕ on T^1M. On the other hand, for every smooth function ϕ on T^1M we have

$$\int \phi(\Delta^s(\phi) + Y(\phi))\, d\eta = -\int \|\nabla^s\phi\|^2\, d\eta \leq 0$$

(see [12]), and consequently $\alpha_\eta < \infty$.

Let $C_c^\infty(\tilde{M})$ be the vector space of smooth functions on $\tilde{M}$ with compact support. Recall that $\delta_0 > 0$ equals the infimum of the Raleigh-quotients of nonvanishing elements of $C_c^\infty(\tilde{M})$. If $\lambda_{\tilde{M}}$ denotes the Lebesgue measure on $\tilde{M}$, then for $\psi \in C_c^\infty(\tilde{M})$ this Rayleigh quotient is just

$$-\int \psi(\Delta\psi)d\lambda_{\tilde{M}} \Big/ \int \psi^2 d\lambda_{\tilde{M}}.$$

Thus it suffices to find a function $\psi \in C_c^\infty(\tilde{M})$ such that for every $\epsilon > 0$

$$\int \psi(\Delta\psi)\, d\lambda_{\tilde{M}} \geq (\alpha_\eta - \epsilon) \int \psi^2\, d\lambda_M.$$

For this we choose $v \in T^1\tilde{M}$ and identify $\tilde{M}$ with $(W^s(v), g^s)$. As before we denote by λ^{ss} the Lebesgue measures on the leaves of the strong stable foliation induced by the Riemannian metric on M, and write $d\lambda^s = dt \times d\lambda^{ss}$ where dt is the 1-dimensional Lebesgue measure on the flow lines of the geodesic flow. We denote moreover by $\nabla\psi$ (resp. $\Delta\psi$) the gradient (resp. Laplacian) of a function ψ on the smooth Riemannian manifold $(W^s(v), g^s)$.

Let $\epsilon > 0$ and choose a smooth function ϕ on T^1M with $\int \phi^2 d\eta = 1$ in such a way that

$$\alpha = \int \phi(\Delta^s(\phi) + Y(\phi) + \phi[\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2])\, d\eta \geq \alpha_\eta - \epsilon.$$

Denote again by ϕ the restriction to $W^s(v)$ of the lift of ϕ to $T^1\tilde{M}$, and choose $c > 0$ sufficiently large that $\|Y\| + |\frac{1}{2}\operatorname{div}(Y) + \|Y\|^2|(w) \leq c$ and

$$[\,\|\nabla^s(\phi^2)\| + \phi^2(1+\|Y\|) + |\phi(\Delta^s\phi + Y(\phi))| + \phi^2|\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2|\,](w) \leq c$$

for every $w \in T^1M$.

Let $\tilde{Y}$ be the lift of Y to $T^1\tilde{M}$, and let f be a positive function on $W^s(v)$ which satisfies $\nabla \log f = \frac{1}{2}\tilde{Y}\,|_{W^s(v)}$. Then f is a function of class C^2, and $\|\nabla f\| + |\Delta(f)| \leq cf$ pointwise on $W^s(v)$.

Let $B_2 \supset B_1$ be compact balls of radius $r_2 > r_1 > 0$ about v in $W^{ss}(v)$, whose boundaries have measure zero with respect to η^{ss} and such that

$$\int_{B_2} f^2\, d\eta^{ss} \leq (1 + \epsilon/2c) \int_{B_1} f^2\, d\eta^{ss}.$$

We then may renormalize f in such a way that $\int_{B_1} f^2\, d\eta^{ss} = 1$.

Choose a smooth Φ^t-invariant function ρ on $W^s(v)$ with values in [0,1] and such that $\rho(w) = 0$ for $w \in W^{ss}(v) - B_2$ and $\rho(w) = 1$ for $w \in B_1$. Since ρ is Φ^t-invariant, there is then a number $t_0 > 0$ such that $|\Delta^s\rho(w)| \leq 1$ and $\|\nabla\rho(w)\| \leq 1$ for every $w \in \bigcup_{t \geq t_0} \Phi^{-t}W^{ss}(v)$. By Proposition 4.1 there is a number $t_1 \geq t_0$ such that for every $t \geq t_1$ the following are satisfied:

$$\int_{\Phi^{-t}B_1} (\phi f^2)(\Delta(\phi) + 2\langle\nabla \log f, \nabla\phi\rangle + \phi[\operatorname{div}(\nabla \log f) + \|\nabla \log f\|^2])\, d\lambda^{ss}$$

$$(1) \qquad = \int_{\Phi^{-t}B_1} (\phi f)\Delta(\phi f)\, d\lambda^{ss} \geq \int_{B_1} f^2\, d\eta^{ss}(\alpha - \epsilon) = \alpha - \epsilon,$$

(2)
$$\int_{\Phi^{-t}(B_2-B_1)} f^2\, d\lambda^{ss} \le \epsilon/c,$$

(3)
$$\int_{\Phi^{-t}B_1} \phi^2 f^2\, d\lambda^{ss} \ge (1+\epsilon)^{-1}.$$

The support of the function $\rho\phi f$ is contained in $\bigcup_{t\in\mathbb{R}} \Phi^t B_2$ and

$$\begin{aligned}|(\rho\phi f)\Delta(\rho\phi f)| \le & f^2[\,|\phi^2\rho\Delta(\rho)| + \rho\|\nabla\rho\|(2\|\phi\nabla\phi\| + \|\tilde{Y}\|\phi^2) \\ & + \rho^2(|\phi(\Delta(\phi)+\tilde{Y}(\phi))| + \phi^2\,|\,\frac{1}{2}\operatorname{div}(\tilde{Y}) + \frac{1}{4}\|\tilde{Y}\|^2|\,)],\end{aligned}$$

and consequently $|(\rho\phi f)\Delta(\rho\phi f)| \le cf^2$ on $\cup_{t\ge t_1}\Phi^{-t}W^{ss}(v)$. Thus for $t \ge t_1$ we obtain

(4)
$$\begin{aligned}&\int_{\Phi^{-t}W^{ss}(v)} (\rho\phi f)\Delta(\rho\phi f)\, d\lambda^{ss} \\ &\quad\ge \int_{\Phi^{-t}B_1} (\phi f)\Delta(\phi f)\, d\lambda^{ss} - \int_{\Phi^{-t}(B_2-B_1)} cf^2\, d\lambda^{ss} \\ &\quad\ge \alpha - 2\epsilon.\end{aligned}$$

Choose a smooth function $\xi\colon \mathbb{R} \to [0,1]$ such that $\xi(t) = 0$ for $t \le 0$, $\xi(t) = 1$ for $t \ge 1$. For an integer $k > 0$, define functions $\xi_k, \zeta_k\colon W^s(v) \to [0,1]$ by $\xi_k(\Phi^t w) = \xi(-t-k)$ and $\zeta_k(\Phi^t w) = \xi(k+t+1)$ for $w \in W^{ss}(v)$ and $t \in \mathbb{R}$. Then the norms of the gradients of ξ_k, ζ_k and the absolute values of $\Delta(\xi_k), \Delta(\zeta_k)$ are pointwise uniformly bounded independent of $k > 0$.

From the above estimates and Proposition 4.1 it then follows:

(5) There is a number $A > 0$ such that

$$\Big|\int_{\Phi^{-t}W^{ss}(v)} (\rho\phi f\zeta_j\xi_k)\Delta(\rho\phi f\zeta_j\xi_k)\, d\lambda^{ss}\Big| \le A$$

for all $j, k \ge 0$ and all $t \ge t_1$.

Choose an integer $m \ge 2A/\epsilon$, let $k > t_1+1$ and define a function ψ on $W^s(v)$ by $\psi = \xi_k\zeta_{m+k}\rho\phi f$. Then ψ is a smooth function with compact support, and $\int_{W^s(v)} \psi(\Delta\psi)\, d\lambda^s = a_1 + a_2 + a_3$ where

$$\begin{aligned}|a_1| &= \Big|\int_{\cup_{t\le k}\Phi^{-t}W^{ss}(v)} \psi(\Delta\psi)\, d\lambda^s\Big| \le A, \\ a_2 &= \int_{\cup_{t=k}^{k+m}\Phi^{-t}W^{ss}(v)} \psi(\Delta\psi)\, d\lambda^s \ge m(\alpha_\eta - 3\epsilon) \quad \text{and} \\ |a_3| &= \Big|\int_{\cup_{t\ge k+m}\Phi^{-t}W^{ss}(v)} \psi(\Delta\psi)\, d\lambda^s\Big| \le A.\end{aligned}$$

Together we obtain that $\int \psi(\Delta\psi)\, d\lambda^s \ge m(\alpha_\eta - 4\epsilon)$, in particular $\alpha_\eta - 4\epsilon < 0$.

On the other hand we have

$$\int \psi^2 \, d\lambda^s \geq \int_{\bigcup_{t=k}^{k+m} \Phi^{-t} B_1} \phi^2 f^2 \, d\lambda^2 \geq m(1+\epsilon)^{-1},$$

and consequently

$$\int \psi(\Delta\psi) \, d\lambda^s / \int \psi^2 \, d\lambda^s \geq (\alpha_\eta - 4\epsilon)(1+\epsilon).$$

Thus also $-\delta_0 \geq (\alpha_\eta - 4\epsilon)(1+\epsilon)$, which implies that $-\delta_0 \geq \alpha_\eta$ since $\epsilon > 0$ was arbitrary.

The next lemma then shows that $\alpha_\eta = -\delta_0$ for every measure η as above:

Lemma 4.3. $-\delta_0 \leq \alpha_\eta$ *for every measure* η *induced as above by the Gibbs-equilibrium state of a Hölder continuous function on* T^1M.

Proof. If suffices to construct a function ϕ on T^1M of class C_s^2 such that $\int \phi^2 \, d\eta = 1$ and $\int \phi(\Delta^s(\phi) + Y(\phi) + \phi[\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2]) \, d\eta \geq -\delta_0 - \epsilon$ for every $\epsilon > 0$.

For this we recall that $-\delta_0$ equals the top of the L^2-spectrum of $\tilde{M}$, and hence for $\epsilon > 0$ there is a compact ball B in $\tilde{M}$ and a smooth function $0 \not\equiv f$ on $\tilde{M}$ with support in B such that

$$-\int f\Delta(f) \, d\lambda_{\tilde{M}} \leq (\delta_0 + \epsilon) \int f^2 \, d\lambda_{\tilde{M}},$$

where $\lambda_{\tilde{M}}$ is the Lebesgue measure on $\tilde{M}$.

Recall that every leaf of the stable foliation of $T^1\tilde{M}$ projects diffeomorphically onto $\tilde{M}$.

Let $\Pi\colon T^1\tilde{M} \to T^1M$ be the canonical projection. If $v \in T^1\tilde{M}$ is such that $\Pi W^s(v)$ does not contain a periodic orbit of the geodesic flow, then the restriction of Π to $W^s(v)$ is injective. This implies that we can find a vector $v \in T^1\tilde{M}$ with $P(v) \in B$, an open neighborhood A of v in $W^s(v)$, an open neighborhood D of v in $W^{su}(v)$ and a homeomorphism Λ of $A \times D$ onto an open neighborhood C of v in $T^1\tilde{M}$ with the following properties:

1) $\Lambda(w, v) = w$ for every $w \in A$.
2) $\Lambda(v, z) = z$ for every $z \in D$.
3) $\Lambda(A \times \{z\})$ is contained in $W^s(z)$ for every $z \in D$ and $P\Lambda(A \times \{z\}) \supset B$.
4) $\Lambda(\{w\} \times D)$ is contained in $W^{su}(w)$ for every $w \in A$.
5) The restriction of Π to C is a diffeomorphism into T^1M.

Recall that the measures η^{su} on the leaves of the strong unstable foliation induce a nonzero measure η^D on D. Denote again by λ^s the family of Lebesgue measures on the manifolds $A \times \{z\} \subset A \times D$ induced via Λ from the Lebesgue measures on the leaves of the stable foliation. Let ρ be the measure on $A \times D$ defined by $d\rho = d\lambda^s \times d\eta^D$. Then Λ is absolutely continuous with respect to the measure ρ on $A \times D$ and the measure η on C. The square root α of the Jacobian of Λ with respect to these measures is Hölder continuous. If $\tilde{Y}$ denotes the lift of the vector field Y to $T^1\tilde{M}$, then $\alpha \circ \Lambda^{-1}$ is of class C_s^2 on C and $\nabla^s \log(\alpha \circ \Lambda^{-1}) = \frac{1}{2}\tilde{Y}$.

Choose a smooth function ψ on D with compact support and values in $[0,1]$ such that $\psi(v) = 1$. Define a function ϕ on C by $\phi(\Lambda(w,z)) = \psi(z)\alpha^{-1}(w,z)f(P(\Lambda(w,z)))$. Then ϕ is a function on C with compact support and hence induces a function $\overline{\phi}$ on T^1M with compact support in $\Pi(C)$. Moreover $\overline{\phi}$ is of class C_s^2.

Write $\overline{\alpha} = \alpha \circ \Lambda^{-1}$ and $\overline{f} = f \circ P$; then

$$\begin{aligned}
\chi &= \int \overline{\phi}(\Delta^s(\overline{\phi}) + Y(\overline{\phi}) + \overline{\phi}[\frac{1}{2}\operatorname{div}(Y) + \frac{1}{4}\|Y\|^2]) \, d\eta \\
&= \int_C \phi(\Delta^s(\phi) + \tilde{Y}(\phi) + \phi[\frac{1}{2}\operatorname{div}(\tilde{Y}) + \frac{1}{4}\|Y\|^2]) \, d\eta \\
&= \int_{A\times D} (\overline{f} \circ \Lambda)\alpha^{-1}[\Delta^s(\overline{f}\overline{\alpha}^{-1}) \circ \Lambda + \tilde{Y}(\overline{f}\overline{\alpha}^{-1}) \circ \Lambda \\
&\quad + (\overline{f} \circ \Lambda)\alpha^{-1}(\frac{1}{2}\operatorname{div}(\tilde{Y}) + \frac{1}{4}\|\tilde{Y}\|^2) \circ \Lambda]\alpha^2\psi^2 \, d\lambda^s \times d\eta^D.
\end{aligned}$$

Now $\nabla^s \log \overline{\alpha} = \frac{1}{2}\tilde{Y}$ and consequently we obtain from the above formula that

$$\begin{aligned}
\chi &= \int_{A\times D} (\overline{f} \circ \Lambda)(\Delta^s(\overline{f}) \circ \Lambda)\psi^2 \, d\lambda^s \times d\eta^D \\
&\geq (-\delta_0 - \epsilon) \int_{A\times B} (\overline{f} \circ \Lambda)^2 \psi^2 \, d\lambda^s \times d\eta^D
\end{aligned}$$

by the choice of $\overline{f}$. But clearly

$$\int \overline{\phi}^2 \, d\overline{\eta} = \int_{A\times D} (\overline{f} \circ \Lambda)^2 \psi^2 \, d\lambda^s \times d\eta^D$$

and therefore $\alpha_\eta \geq -\delta_0 - \epsilon$ by the definition of α_η. Since $\epsilon > 0$ was arbitrary, the lemma follows.

Recall that the Lebesgue Liouville measure λ on T^1M is the Gibbs equilibrium state of the Hölder continuous function $v \to tr\, U(v)$ where $tr\, U(v)$ is the trace of the second fundamental form at Pv of the horsphere $PW^{su}(v)$. Denote the g^s-gradient of λ by Z. Then we have:

Lemma 4.4. *The differential operator $L = \Delta^s + Z + \frac{1}{2}\operatorname{div}(Z) + \frac{1}{4}\|Z\|^2$ is self-adjoint with respect to λ, and the top of its spectrum equals δ_0.*

Proof. Since Z is the g^s-gradient of λ, the operator L is self-adjoint with respect to λ by Corollary 2.6 of [12].

Let Δ^v be the leafwise Laplacean of the vertical foliation, i.e., for a smooth function f on T^1M and every $v \in T^1M$ the evaluation of Δ^v on f at v is obtained by restricting f to the fibre $T^1_{Pv}\tilde{M}$ of the fibration $T^1M \to M$ through v and evaluating the Laplacean of the round sphere $T^1_{Pv}M$ on this restriction. Then Δ^v is a second order differential operator on T^1M with smooth coefficients, which is subordinate to the vertical foliation and leafwise elliptic. Moreover Δ^v is self-adjoint with respect to the invariant measure λ, i.e., for smooth functions f, ϕ on T^1M we have $\int f(\Delta^v\phi)\, d\lambda = \int \phi(\Delta^v f)\, d\lambda = -\int \langle \nabla^v f, \nabla^v \phi\rangle\, d\lambda$ where $\nabla^v f$ is the section of the vertical bundle T^v whose restriction to a fibre T^1_pM equals the gradient of the restriction of f to the (totally geodesic) submanifold T^1_pM of T^1M, and by abuse of notation $\langle , \rangle$ is the natural Riemannian metric on T^v.

Since the vertical foliation and the stable foliation of T^1M are transversal, for every $\epsilon > 0$ the operator $L_\epsilon = L + \epsilon\Delta^v$ is elliptic and moreover self-adjoint with respect to λ. In particular the spectrum of L_ϵ is a pure point spectrum, and its top is an eigenvalue α_ϵ whose corresponding eigenspace is one-dimensional and spanned by a positive function $f_\epsilon\colon T^1M \to (0,\infty)$ of class C^2. We assume f_ϵ to be normalized in such a way that $\int f_\epsilon\, d\lambda = 1$. First we note:

Lemma 4.5. $\lim_{\epsilon\to 0} \alpha_\epsilon = -\delta_0$.

Proof. Let Q_ϵ be the quadratic form on the space of smooth functions on T^1M associated to L_ϵ; for every smooth function ϕ on T^1M we have

$$Q_\epsilon(\phi) = \int \phi(L_\epsilon\phi)\, d\lambda = \int \phi(L\phi)\, d\lambda - \epsilon \int \|\nabla^v\phi\|^2\, d\lambda,$$

and consequently $Q_\epsilon \geq Q_\delta$ for $\epsilon \leq \delta$. Now the space of smooth functions on T^1M is a form core for the quadratic form Q_0 defined by L; since $Q_\epsilon \to Q_0 (\epsilon \to 0)$ on this form core, the operators L_ϵ converge as $\epsilon \to 0$ in the strong resolvent sense to L (see [6]).

This implies in particular that $\lim_{\epsilon\to 0} \alpha_\epsilon = -\delta_0$.

Lemma 4.6. *Let η be a weak limit of the measures $f_\epsilon\lambda$ on T^1M as $\epsilon \to 0$. Then η is a harmonic measure for the operator $L + \delta_0$.*

Proof. Let ϕ be a smooth function on T^1M; then ϕ and $\Delta^v\phi$ are continuous. Hence $\int \epsilon(\Delta^v\phi) f_\epsilon\, d\lambda \to 0$ and

$$(\alpha_\epsilon + \delta_0) \int \phi f_\epsilon\, d\lambda \to 0 \ (\epsilon \to 0)$$

by Lemma 4.5. Let $\{\epsilon_i\}_i$ be a sequence such that $\epsilon_i \to 0$ and that the measures $f_{\epsilon_i}\lambda$ converge weakly as $i \to \infty$ to a measure η. We then have

$$\begin{aligned}\int (L+\delta_0)\phi\, d\eta &= \lim_{i\to\infty}\int [(L+\delta_0)\phi] f_{\epsilon_i}\, d\lambda \\ &= \lim_{i\to\infty}\int [(L+\epsilon_i\Delta^v - \alpha_{\epsilon_i})\phi] f_{\epsilon_i}\, d\lambda \\ &= \lim_{i\to\infty}\int \phi(L_{\epsilon_i}-\alpha_{\epsilon_i})(f_{\epsilon_i})\, d\lambda = 0,\end{aligned}$$

since L_{ϵ_i} is self-adjoint with respect to λ. This shows the lemma.

Corollary 4.7. *Let η be as in Lemma 4.6, and let ζ be the section of TW^s such that $\zeta + \frac{1}{2}Z$ is the g^s-gradient of η. Then*

$$\operatorname{div}(\zeta) + \|\zeta\|^2 + \delta_0 = 0.$$

Proof. Let $v \in T^1\tilde{M}$ and let f be a function on $W^s(v)$ such that $\nabla^s \log f = \frac{1}{2}Z\,|_{W^s(v)}$. For a smooth function ϕ on $W^s(v)$ with compact support we then have $f^{-1}\Delta^s(f\phi) = \Delta^s(\phi)+Z(\phi)+\phi f^{-1}\Delta(f) = L\phi$, and hence the formal adjoint L^* of $L_{|W^s(v)}$ is given by $L^*(\phi) = f\Delta^s(f^{-1}\phi)$. In other words, if $L^*(\phi) = -\delta_0\phi$, then $f^{-1}\phi$ is a solution of $\Delta^s(f^{-1}\phi) = -\delta_0 f^{-1}\phi$.

From this and Lemma 2.2 of [12] the corollary follows.

5. Pressure computation

In this section we use the results in Section 4 to prove the second part of Theorem B and Theorem C. For this we continue to use the assumptions and notation of Sections 1-4. Recall in particular that we denoted the pressure of the functions $2\langle X, \xi_\epsilon\rangle$ for $\epsilon \in (0,\delta_0]$ by $q(\epsilon) < 0$. Our theorem will be a consequence of the fact that $\lim_{\epsilon\to 0} q(\epsilon) = 0$. As in Section 4 let $L_\delta = \Delta^s + Z + \frac{1}{2}\operatorname{div}(Z) + \frac{1}{4}\|Z\|^2 + \delta\Delta^v$, and let f_δ be an eigenfunction of L_δ with respect to the largest eigenvalue α_δ. In contrast to Section 4 however we assume now that f_δ is normalized in such a way that $\int f_\delta^2 d\lambda = 1$. Then we have:

Lemma 5.1. *Let ν be a weak limit of the measures $f_\delta^2\lambda$ on T^1M as $\delta \to 0$. Then the following are satisfied:*

i) *The vector fields ξ_ϵ converge as $\epsilon \to 0$ in the Hilbert space of sections of TW^s over T^1M, which are square integrable with respect to ν to a section ξ of TW^s.*
ii) $\operatorname{div}(\xi) + \|\xi\|^2 + \delta_0 = 0$ *almost everywhere on (T^1M,ν).*
iii) *ν is a self-adjoint harmonic measure for $\Delta^s + 2\xi$.*

iv) *Every ν-measurable section ζ of TW^s over T^1M, which satisfies* $\operatorname{div}(\zeta) + \|\zeta\|^2 + \delta_0 \leq 0$ *almost everywhere, coincides with* ξ.

Proof. Let $\{\delta_i\}_i$ be a sequence such that $\delta_i \to 0$ $(i \to \infty)$ and that the measures $f_{\delta_i}^2 \lambda$ converge as $i \to \infty$ weakly to a measure ν. For $i > 0$ write $f_i = f_{\delta_i}, \alpha_i = \alpha_{\delta_i}$ and $Q_i = \nabla^s \log f_i + \frac{1}{2} Z$. The differential equation for f_i then yields

$$\operatorname{div}(Q_i) + \|Q_i\|^2 - \alpha_i + \delta_i f_i^{-1} \Delta^v(f_i) = 0, \tag{1}$$

and consequently

$$\operatorname{div}(\xi_\epsilon - Q_i) = \|Q_i\|^2 - \|\xi_\epsilon\|^2 - \delta_0 + \epsilon - \alpha_i + \delta_i f_i^{-1} \Delta^v(f_i) \tag{2}$$

for every $\epsilon > 0$. Since $f_i^2 \lambda$ is a self-adjoint harmonic measure for $\Delta^s + 2Q_i$ (see [12]), integration of equation (2) shows

$$\begin{aligned} 0 &= \int (\operatorname{div}(\xi_\epsilon - Q_i) + 2\langle Q_i, \xi_\epsilon - Q_i \rangle) f_i^2 \, d\lambda \\ &= \int (-\|\xi_\epsilon - Q_i\|^2 - \delta_0 + \epsilon - \alpha_i - \delta_i \|\nabla^v \log f_i\|^2) f_i^2 \, d\lambda, \end{aligned}$$

since $\int (f_i^{-1} \Delta^v(f_i)) f_i^2 \, d\lambda = -\int \|\nabla^v \log f_i\|^2 f_i^2 \, d\lambda$ by self-adjointness of Δ^v. From this we obtain

$$\limsup_{i \to \infty} \int \|\xi_\epsilon - Q_i\|^2 f_i^2 \, d\lambda \leq \epsilon. \tag{3}$$

Since the above equation is valid for every $\epsilon > 0$ we further conclude that

$$\limsup_{i \to \infty} \delta_i \int \|\nabla^v \log f_i\|^2 f_i^2 \, d\lambda = 0. \tag{4}$$

Now by the definition of ν we have

$$\begin{aligned} \int \|\xi_\epsilon - \xi_\delta\|^2 \, d\nu &= \lim_{i \to \infty} \int \|\xi_\epsilon - \xi_\delta\|^2 f_i^2 \, d\lambda \\ &\leq \limsup_{i \to \infty} 2\Big(\int \|\xi_\epsilon - Q_i\|^2 f_i^2 \, d\lambda + \int \|\xi_\delta - Q_i\|^2 f_i^2 \, d\lambda\Big) \\ &= 2\epsilon + 2\delta \end{aligned}$$

by the above estimates for all $\epsilon, \delta > 0$. Hence for every sequence $\{\epsilon_j\}_{j>0}$ with $\epsilon_j \to 0$ $(j \to \infty)$ the vector fields $\{\xi_{\epsilon_j}\}_j$ form a Cauchy sequence in the Hilbert space $\mathcal{H}$ of sections of TW^s over T^1M, which are square integrable with respect to ν. In other words, there is a section $\xi \in \mathcal{H}$ such that $\xi_\delta \to \xi$ $(\delta \to 0)$ in $\mathcal{H}$ which yields i) above.

Next we want to show that ν is a self-adjoint harmonic measure for $\Delta^s + 2\xi$, and for this it is sufficient to show that

$$\int(\operatorname{div}(Y) + \langle 2\xi, Y\rangle)\ d\nu = 0$$

for every section Y of TW^s of class C^1_s. Let Y be a section of TW^s of class C^1_s and let $\epsilon > 0$; since $\xi_\delta \to \xi$ in $\mathcal{H}$ there is a number $\delta \leq \epsilon$ such that

$$|\int\langle 2\xi, Y\rangle\ d\nu - \int\langle 2\xi_\delta, Y\rangle\ d\nu\ | < \epsilon. \tag{5}$$

Now the functions $\langle 2\xi_\delta, Y\rangle$ and $\operatorname{div}(Y)$ are continuous on T^1M and the measures $f_i^2\lambda$ converge as $i \to \infty$ weakly to ν. This means that we can find a number $i_0 > 0$ such that

$$|\int(\operatorname{div}(Y) + \langle 2\xi_\delta, Y\rangle)\ d\nu - \int(\operatorname{div}(Y) + \langle 2\xi_\delta, Y\rangle)f_i^2\ d\lambda\ | < \epsilon \tag{6}$$

for all $i > i_0$. On the other hand, by (4) above we may further assume that

$$|\ \delta_i \int f_i \Delta^v(f_i)\ d\lambda - \alpha_i - \delta_0\ | < \epsilon \tag{7}$$

for all $i > i_0$. The equation preceding (3) then implies that $\int \|\xi_\delta - Q_i\|^2 f_i^2\ d\lambda \leq 2\epsilon$ so that

$$|\int\langle 2\xi_\delta, Y\rangle f_i^2\ d\lambda - \int\langle 2Q_i, Y\rangle f_i^2\ d\lambda\ | \leq 2c\sqrt{2\epsilon}, \tag{8}$$

where $c = \max\{\|Y\|(v) \mid v \in T^1M\}$.

Since $f_i^2\ d\lambda$ is a self-adjoint harmonic measure for $\Delta^s + 2Q_i$, integration and (6), (7), (8) yield

$$\begin{aligned} |\int(\operatorname{div}(Y) + \langle 2\xi, Y\rangle)d\nu\ | &\leq 2\epsilon + 2c\sqrt{2\epsilon} + |\int(\ \operatorname{div}(Y) + \langle 2Q_i, Y\rangle)f_i^2\ d\lambda\ | \\ &= 2(\epsilon + c\sqrt{2\epsilon}). \end{aligned}$$

Since $\epsilon > 0$ was arbitrary we obtain that indeed

$$\int(\ \operatorname{div}(Y) + \langle 2\xi, Y\rangle)\ d\nu = 0,$$

and hence iii).

Now ν is a self-adjoint harmonic measure for a leafwise elliptic second order differential operator subordinate to W^s, and hence ν is absolutely continuous with respect to the stable and strong unstable foliation, with conditionals on stable manifolds in the Lebesgue measure class. But this means that for ν-almost every $v \in T^1M$ the restriction of the vector fields ξ_δ to the open ball B of radius 1 about v in $W^s(v)$ converge almost everywhere pointwise with respect to the Lebesgue measure λ^s on $W^s(v)$ to the restriction of ξ by i) above, and $\|\xi_\delta\|^2 \to \|\xi\|^2$ almost everywhere pointwise on $(W^s(v), \lambda^s)$ as well. But $\mathrm{div}(\xi_\delta) + \|\xi_\delta\|^2 + \delta_0 - \delta = 0$ and consequently via partial integration we obtain that $\mathrm{div}(\xi) + \|\xi\|^2 + \delta_0 = 0$ on B in the sense of distributions. Regularity theory for elliptic equations then implies that in fact the restriction of ξ to B is a strong solution of $\mathrm{div}(\xi) + \|\xi\|^2 + \delta_0 = 0$ and hence $\mathrm{div}(\xi) + \|\xi\|^2 + \delta_0 = 0$ almost everywhere with respect to ν.

We are left with statement iv) in the lemma. For this let χ be any ν-measurable square integrable section of TW^s over T^1M, which satisfies $\mathrm{div}(\chi) + \|\chi\|^2 + \delta_0 \leq 0$ almost everywhere with respect to ν. As before we then have

$$\begin{aligned} 0 &\geq \int (\mathrm{div}(\chi - \xi) + \|\chi\|^2 - \|\xi\|^2)\, d\nu \\ &= \int (\langle 2\xi, \xi - \chi\rangle + \|\chi\|^2 - \|\xi\|^2)\, d\nu \\ &= \int \|\xi - \chi\|^2\, d\nu, \end{aligned}$$

since ν is a self-adjoint harmonic measure for $\Delta^s + 2\xi$. Hence $\xi = \chi$ almost everywhere.

By Lemma 5.1 iii) the measure ν is harmonic for the leafwise elliptic differential operator $\Delta^s + 2\xi$. Therefore by the result of Garnett [8] we can write $d\nu = d\lambda^s \times d\nu^{su}$ where ν^{su} is a family of locally finite Borel-measures on the leaves of W^{su}, which are absolutely continuous under canonical maps, and where λ^s is the family of Lebesgue measures on the leaves of W^s for all $\epsilon > 0$.

In other words, the measures ν^{su} induce a $\pi_1(M)$-invariant measure class $\nu(\infty)$ on $\partial\tilde{M}$. This measure class has the properties mentioned in Theorem C:

Corollary 5.2. *For every $x \in \tilde{M}$ and $\nu(\infty)$-almost every $\zeta \in \partial\tilde{M}$ the functions $y \to K_\epsilon(x, y, \zeta)$ converge as $\epsilon \to 0$ uniformly on compact subsets of $\tilde{M}$ to a minimal positive Δ_0-harmonic function.*

Proof. Let $\tilde{\nu}$ be the lift of ν to a locally finite measure on $T^1\tilde{M}$, and let $\tilde{\xi}$ be the lift of ξ. Then Lemma 5.1 implies that for $\tilde{\nu}$-almost every $v \in T^1\tilde{M}$ the functions $y \to K_\epsilon(x, y, \pi(v))$ converge as $\epsilon \to 0$ uniformly on compact subsets of $\tilde{M}$ to a positive Δ_0-harmonic function f^v. The gradient of $\log f^v$ is just the projection to $\tilde{M}$ of the restriction of $\tilde{\xi}$ to $W^s(v)$.

We are left with showing that for $\tilde{\nu}$-almost every $v \in T^1\tilde{M}$ the function f^v is in fact minimal Δ_0-harmonic. Since for every smooth function

ϕ on $\tilde{M}$ we have

$$f_v^{-1}\Delta(\phi f^v) + \delta_0\phi = \Delta(\phi) + 2\langle\nabla \log f^v, \nabla\phi\rangle,$$

this is equivalent to saying that every bounded $\Delta + 2\nabla \log f^v$-harmonic function on $\tilde{M}$ is constant. Now ν is a self-adjoint harmonic measure for $\Delta^s + 2\xi$, and hence the Kaimanovich-entropy of the diffusion on T^1M induced by $(\Delta^s + 2\xi, \nu)$ vanishes (see [12], [15]). But this just means that ν-almost every leaf of W^s is Liouville with respect to $\Delta^s + 2\xi$, which yields the corollary.

Consider now again the measures ν^{su} on the leaves of the strong unstable foliation. The arguments in the proof of Lemma 3.5 then show that there is a number $c > 0$ such that $\nu^{su}(B^{su}(v,1)) \in [c^{-1}, c]$ for all $v \in T^1M$, where $B^i(v,\delta)$ denotes the open ball of radius $\delta > 0$ about v in the manifold $W^i(v)$ equipped with the metric g^i which is induced from the Riemannian metric on M $(i = s, su, ss)$.

Recall that the unique Gibbs equilibrium state ν_ϵ of the function $2\langle X, \xi_\epsilon\rangle$ admits a family ν_ϵ^{su} of conditional measures on strong unstable manifolds such that $\frac{d}{dt}\nu_\epsilon^{su} \circ \Phi^t \mid_{t=0} = 2\langle X, \xi_\epsilon\rangle + q(\epsilon)$. By the arguments in the proof of Lemma 2.7 we have $\nu_\epsilon^{su}(B^{su}(v,1)) \in [c^{-1}, c]$ for all $v \in T^1M$ independent of ϵ. Let $\mathcal{F}\colon v \to -v$ be the *flip* on T^1M and define for $\epsilon > 0$ a measure ν_ϵ^s on the leaves of W^s by $d\nu_\epsilon^s = dt \times d\nu_\epsilon^{ss}$ where $\nu_\epsilon^{ss} = \nu_\epsilon^{su} \circ \mathcal{F}$. Clearly there is a number $a > 0$ such that $\nu_\epsilon^s(B^s(v,1)) \in [a^{-1}, a]$ for all $v \in T^1M$ and all $\epsilon \in (0, \delta_0]$. Thus we obtain a finite Borel measure σ_ϵ on T^1M by defining $d\sigma_\epsilon = d\nu_\epsilon^s \times d\nu^{su}$ which we may assume to be normalized in such a way that $\sigma_\epsilon(T^1M) = 1$ for all $\epsilon > 0$. Then the section ξ of TW^s over T^1M is contained in the Hilbert space of sections which are square integrable with respect to σ_ϵ for all $\epsilon > 0$, with Hilbert norm bounded independent of ϵ. Moreover σ_ϵ is quasi-invariant under the action of the geodesic flow, and we have $\frac{d}{dt}\sigma_\epsilon \circ \Phi^t \mid_{t=0} (v) = 2\langle X, \xi\rangle(v) - 2\langle X, \xi_\epsilon\rangle(-v) - q(\epsilon)$ where as before $q(\epsilon) < 0$ is the pressure of the function $2\langle X, \xi_\epsilon\rangle$ on T^1M.

Lemma 5.3. *For every $\delta > 0$ there is a number $\epsilon(\delta) > 0$ such that $\int \|\xi_\epsilon - \xi\|^2 d\sigma_\epsilon < \delta$ for all $\epsilon < \epsilon(\delta)$.*

Proof. Recall that the vector fields ξ_ϵ, ξ are pointwise uniformly bounded in norm, independent of ϵ. Lemma 5.1 together with the precompactness of the space of positive locally bounded Δ_ϵ-harmonic functions on $\tilde{M}$ then implies the following: Let $\tilde{\nu}^{su}$ be the lift of the measures ν^{su} to the leaves of $W^{su} \subset T^1\tilde{M}$. Then for every $v \in T^1\tilde{M}$ and $\tilde{\nu}^{su}$-almost every $w \in W^{su}(v)$ the restriction of $\tilde{\xi}_\epsilon$ to $W^s(w)$ converges uniformly on compact sets to the restriction of ξ.

Let $C \subset T^1\tilde{M}$ be a set with a local product structure, given by a vector $v \in T^1\tilde{M}$, a compact ball $B \subset W^{su}(v)$ about v, a compact ball $A \subset W^s(v)$ about v and a homeomorphism $\Lambda\colon A \times B \to C$ such that $\Lambda(w,z) \in W^s(z) \cap W^{su}(w)$ as in the proof of Lemma 4.3. We assume that the projection of C to T^1M is surjective.

Since C can be covered by a finite number of fundamental domains for the action of $\pi_1(M)$ on $T^1\tilde{M}$, there is a number $c_0 > 0$ such that $\sigma_\epsilon(C) \le c_0$ for all $\epsilon \in (0, \delta_0]$, where we denote the lift of σ_ϵ to $T^1\tilde{M}$ again by σ_ϵ. By the infinitesimal Harnack inequality we can further choose a number $m > 0$ such that $\|\xi_\epsilon\|^2(v)$ and $\|\xi\|^2(v)$ is not larger than m for all $v \in T^1M$ and all $\epsilon \in (0, \delta_0]$.

Let $\delta > 0$ be given. By the properties of the measures ν_ϵ^s there is then a number $\rho > 0$ such that $\sigma_\epsilon(\Lambda(A \times E)) < \delta/8m$ whenever $E \subset B$ is Borel and $\tilde{\nu}^{su}(E) < \rho$. On the other hand, for $\tilde{\nu}^{su}$-almost every $w \in B$ the sections ξ_ϵ converge on $\Lambda(A \times \{w\})$ uniformly to ξ as $\epsilon \to 0$; hence there is a number $\epsilon(\delta) > 0$ such that $\tilde{\nu}^{su}(E) < \rho$ where $E = \{w \in B \mid \|\xi_\epsilon - \xi\|^2(\Lambda(z,w)) \ge \delta/2c_0 \text{ for some } z \in A \text{ and } \epsilon \le \epsilon(\delta)\}$.

For $\epsilon < \epsilon(\delta)$ we then have

$$\begin{aligned}\int \|\xi_\epsilon - \xi\|^2 d\sigma_\epsilon \le & \int_C \|\xi_\epsilon - \xi\|^2 d\sigma_\epsilon \\ = & \int_{\Lambda(A\times E)} \|\xi_\epsilon - \xi\|^2 d\sigma_\epsilon + \int_{\Lambda(A\times(B-E))} \|\xi_\epsilon - \xi\|^2 d\sigma_\epsilon \\ \le & 4m\sigma_\epsilon(\Lambda(A \times E)) + \sigma_\epsilon(\Lambda(A \times B))\delta/2c_0 \le \delta\end{aligned}$$

by the above. This shows the lemma.

Corollary 5.4. $q(0) = \lim_{\epsilon\to 0} q(\epsilon) = 0$.

Proof. Assume to the contrary that $q(0) = \lim_{\epsilon\to 0} q(\epsilon) < 0$; recall that $q(\epsilon) < q(0)$ for every $\epsilon > 0$. By Lemma 5.3 we then can find a number $\epsilon > 0$ such that $\int \|\xi_\epsilon - \xi\|^2 d\sigma_\epsilon < \frac{1}{16} q(0)^2$. Since the norm of the geodesic spray X is constant 1, from this it follows that

$$\left| \int \langle X, \xi - \xi_\epsilon \rangle d\sigma_\epsilon \right| \le \int \|\xi - \xi_\epsilon\| d\sigma_\epsilon \le \left(\int \|\xi - \xi_\epsilon\|^2 d\sigma_\epsilon\right)^{1/2} < -\frac{1}{4} q(0).$$

But $\frac{d}{dt}\sigma_\epsilon \circ \Phi^t \mid_{t=0} = 2\langle X, \xi - \xi_\epsilon\rangle - q(\epsilon)$ and consequently

$$0 = \int \frac{d}{dt}\sigma_\epsilon \circ \Phi^t \mid_{t=0} d\sigma_\epsilon = \int 2\langle X, \xi - \xi_\epsilon \rangle d\sigma_\epsilon - q(\epsilon) \ge -\frac{1}{2} q(0)$$

by the above estimates, a contradiction to our assumption $q(0) < 0$. Hence the corollary is proved.

As a corollary we obtain the second part of Theorem B.

Corollary 5.5.

1) *There is a number $c > 0$ such that $\int_{S(p,R)} G_0(p,y)^2 d\lambda_{p,R}(y) \le c$ for all $p \in \tilde{M}$, all $R \ge 1$.*
2) $\liminf_{R\to\infty} \int_{S(p,R)} G_0(p,y)^{2-\epsilon} d\lambda_{p,R} = \infty$ *for every $\epsilon > 0$.*

Proof. Statement 1) follows from the arguments in the proof of Corollary 3.6. To show 2) let $\epsilon > 0$; by the first part of Theorem B there is then a number $\alpha > 0$ such that $G_0(p,y)^{2-\epsilon} \ge \alpha^{-1} e^{-\alpha \,\mathrm{dist}(p,y)} G_0(p,y)^2$ for all $y, p \in \tilde{M}$ with $\mathrm{dist}(p,y) \ge 1$. Choose now $\epsilon > 0$ sufficiently small that $q(\epsilon) > -\alpha/2$; such a number exists by Corollary 5.3. The Harnack-inequality at infinity of Ancona for the operator Δ_ϵ implies that there is a number $c(\epsilon) > 0$ such that $\int_{S(p,R)} G_\epsilon(p,y)^2 e^{-q(\epsilon)R} d\lambda_{p,R}(y) \ge c(\epsilon)$ for all $R \ge 1$. But the maximum principle yields that $G_0(p,y) \ge \bar{c} G_\epsilon(p,y)$ for all $p, y \in \tilde{M}$ with $\mathrm{dist}(p,y) \ge 1$, where $\bar{c} > 0$ is a universal constant. Hence

$$\begin{aligned}\int_{S(p,R)} G_0(p,y)^{2-\epsilon} d\lambda_{p,R}(y) \ge & \alpha^{-1}\bar{c} \int_{S(p,R)} G_\epsilon(p,y)^2 e^{-\alpha R} d\lambda_{p,R}(y) \\ \ge & \alpha^{-1}\bar{c} c(\epsilon) e^{\alpha R/2}\end{aligned}$$

for all $R \ge 1$, and the corollary is proved.

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